On the Accuracy of Runge-Kutta Methods for Unsteady Linear Wave Equation

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Abstract

The von Neumann analysis is carried out to study the dissipation, dispersion and stability limits of the unsteady linear wave equation solved by the standard 4-stage Runge-Kutta method with several widely used spacial differencing schemes, including 2nd order central differencing, 2nd order upwind, 3rd order and 4th order biased upwind, and 4th order central differencing. The 2nd order Lax-Wendroff scheme and the 2-stage Runge-Kutta method are also analyzed as references. For a central differencing with the 4-stage Runge-Kutta method, there is a CFL limit, under which the solution is dissipation free. The dissipation free CFL limit is far below the stability CFL limit. There is also a CFL limit under which the dispersion error of a central differencing scheme is independent of CFL number. The dispersion error exists for all the schemes studied. The numerical results indicated that the dissipation and dispersion error of upwind schemes with 4-stage Runge-Kutta method are independent of the CFL number under the CFL stability limit. For the wave equation with a low frequency solution studied in this paper, the 4th order central differencing and the 4th order biased upwind differencing have similar level of accuracy.

1 Introduction

Since Runge-Kutta methods were introduced to solve the steady state solutions governed by the time dependent Euler equations in early 1980's[1], it has been widely applied to time accurate unsteady flow simulations to achieve high order temporal accuracy. For example, the 4-stage Runge-Kutta methods with higher than second order temporal accuracy are usually chosen for simulation of turbulence (e.g. DNS and LES) and aeroacoustics[2].

However, there is a lack of systematic study about the behavior of Runge-Kutta methods for their dissipation and dispersion errors for time accurate calculation. Our knowledge is still mostly at the level of 1980's. For example, we know that the 4-stage Runge-Kutta method with 2nd order central differencing has better stability condition than most of other explicit schemes with CFL stability limit of $2\sqrt{2}$, [3][4][5][1]. However, there are two questions that have no clear answers: 1) Is it dissipation free under this CFL stability limit?; 2) What is the behavior of the dispersion error?

The stability analysis of the 4-stage Runge-Kutta method given in [3][4][5] is based on the nature of the ordinary differential equations. It does not provide the dissipation and dispersion information as the von Neumann analysis does, which is critical to unsteady CFD calculation. In addition, the stability limit of $2\sqrt{2}$ is for the 4-stage Runge-Kutta method with the 2nd order central differencing. The stability limits of the 4-stage Runge-Kutta method with the other schemes studied in this paper were not known. The advantage of the von Neumann analysis is that it uses the hyperbolic linear wave equation that is the CFD model equation, and it gives the information of stability, dissipation, and dispersion simultaneously.

For steady state solutions, the stability limit is the primary interest to achieve fast convergence. The dissipation during the time marching to the converged solution is not a concern since the accuracy of the final solutions are controlled by the spatial accuracy. If a central differencing is used, the spatial scheme has no dissipation according to the Taylor series. If an upwind scheme is used, the spatial scheme will have some inherent dissi-

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pation. The artificial or inherent dissipation for steady state solutions is to increase the stability of the computation and smoothen the discontinuities such as shock waves. Hence, as long as a numerical algorithm can make the steady state solutions converge, the temporal accuracy of the 1st order or the 4th order does not matter. The dispersion error is generally not an issue for steady state solutions since waves are stationary. The place that the dispersion can affect the steady state solution is the monotonicity of the discontinuities.

For unsteady calculation, understanding of the dissipation and dispersion error is crucial. The dissipation will smear the wave amplitude and the dispersion will shift the phase of the waves. Both dissipation and dispersion are harmful to an unsteady solution. A dissipation free scheme in space does not necessarily lead to a dissipation free solution in time. For some central differencing schemes, it is possible to achieve dissipation free solution in time. However, if the dispersion error is large, the solution with correct wave amplitude and wrong phase can be completely meaningless.

Unlike other explicit schemes such as the Euler explicit method, Lax-Wendroff scheme, Leap Frog Scheme, etc., the behavior of the 4-stage Runge-Kutta scheme for its temporal accuracy is not well understood. Reference [2] indicates that the 4-stage Runge-Kutta method with a 6th order central differencing scheme(*Padé* scheme) have a dissipative character, even though the spatial discretization is not dissipative. However, no systematic results on the time accuracy of the 4-stage Runge-Kutta methods have been reported.

The purpose of this paper is to use the von Neumann analysis to study the dissipation, dispersion and stability limits of the 4-stage Runge-Kutta methods for unsteady linear wave equation with some popularly used spatial differencing schemes, including 2nd order central differencing, 2nd order upwind, 3rd and 4th order biased upwind, and 4th order central differencing. These results may be used as the references to develop new discretization schemes for CFD. It appears that this paper is the first work to carry out the von Neumann analysis for the schemes mentioned above.

Runge-Kutta methods can have infinite number of different formulations. The focus here is on the standard Runge-Kutta methods, i.e. standard 2-stage and 4-stage methods.

2 Wave Equation and the Standard Runge-Kutta Methods

2.1 Wave Equation

The wave equation to be studied is the linear wave equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{1}$$

where u can be considered as the convection flow velocity, c the wave speed, t is the time and x is the coordinate.

In order to use Runge-Kutta methods, this equation can be re-written in a pseudo partial differential equation form:

$$\frac{du}{dt} = R(u) \tag{2}$$

where

$$R(u) = -c\frac{\partial u}{\partial x} \tag{3}$$

Eq.(2) can then be marched in the temporal direction using Runge-Kutta methods as the integration for a partial differential equation. The right hand side $R(u)$ can be discretized using different spatial discretization schemes. In this paper, the linear wave equation, eq.(1), is solved with $c = 1$.

2.2 Runge-Kutta Methods (R-K)

Runge-Kutta methods are to construct a single step algorithm with multi-stages, which is independent of the function $R(u)$. The k-stage and k-th order Runge-Kutta methods should satisfy the following Taylor-series expansion up to the k-th derivative term:

$$\begin{aligned} u^{n+1} = u^n + \Delta t u_t^n + \frac{(\Delta t)^2}{2} u_{tt}^n \\ + \frac{(\Delta t)^3}{6} u_{ttt}^n + \frac{(\Delta t)^4}{24} u_{tttt}^n + \ldots \end{aligned} \tag{4}$$

For example, the 2-stage 2nd order Runge-Kutta method should match the Taylor-series expansion up to the term $\frac{(\Delta t)^2}{2} u_{tt}^n$, and the 4-stage 4th order method should match to the term $\frac{(\Delta t)^4}{24} u_{tttt}^n$.

The Lax-Wendroff type schemes [6] are also based on the same Taylor-series expansion, eq.(4). For the Lax-Wendroff scheme, the time derivatives are replaced by the spatial derivatives through the governing equation, eq. (1). In other words, the Runge-Kutta methods are equivalent to Lax-Wendroff type schemes. The difference is that the Lax-Wendroff type schemes realize the Taylor-series in one stage, while Runge-Kutta methods achieve it through multi-stages. Therefore, the Lax-Wendroff type schemes can be obtained via Runge-Kutta methods using certain spatial discretization. Compared with the Lax-Wendroff type schemes, the advantage of Runge-Kutta schemes is that it is easier to be implemented, in particular for the high order accuracy multi-dimensional calculations[7]

2.2.1 Standard two stage, 2nd order Runge-Kutta scheme

Stage 1:

$$u^{(1)} = u^n + \Delta t R^{(n)} \tag{5}$$

Stage 2:

$$u^{n+1} = u^n + \frac{\Delta t}{2}(R^{(n)} + R^{(1)}) \tag{6}$$

where

$$R^{(n)} = R(u^n) = -c\frac{\partial u^n}{\partial x} \tag{7}$$

2.2.2 Standard four stage, 4th order Runge-Kutta scheme

Stage 1:

$$u^{(1)} = u^n + \frac{\Delta t}{2} R^{(n)} \tag{8}$$

Stage 2:

$$u^{(2)} = u^n + \frac{\Delta t}{2} R^{(1)} \tag{9}$$

Stage 3:

$$u^{(3)} = u^n + \Delta t R^{(2)} \tag{10}$$

Stage 4:

$$u^{n+1} = u^n + \frac{\Delta t}{6}(R^{(n)} + 2R^{(1)} + 2R^{(2)} + R^{(3)}) \tag{11}$$

where $R^{(n)}$ is evaluated as eq.(7).

2.2.3 The Lax-Wendroff Scheme

The 2nd order Lax-Wendroff scheme[6] may be considered as a representative of the explicit schemes for hyperbolic equations. The analysis of Lax-Wendroff scheme is given here for the purpose of comparison with Runge-Kutta methods.

Using the wave equation, eq.(1), the time derivatives in the Taylor-series expansion eq.(4) can be replaced by the following spatial derivatives:

$$\begin{aligned} u_t = -cu_x, u_{tt} = c^2 u_{xx}, \\ u_{ttt} = -c^3 u_{xxx}, u_{tttt} = c^4 u_{xxxx} \end{aligned} \tag{12}$$

Use the same Taylor-series expansion for the Runge-Kutta method, eq.(4), we have:

$$\begin{aligned} u^{n+1} = u^n - c\Delta t u_x^n + \frac{(c\Delta t)^2}{2} u_{xx}^n \\ -\frac{(c\Delta t)^3}{6} u_{xxx}^n + \frac{(c\Delta t)^4}{24} u_{xxxx}^n + ... \end{aligned} \tag{13}$$

This is the Lax-Wendroff type scheme, which can achieve any order of accuracy in one step. The 2nd order Lax-Wendroff scheme can be constructed by using the first three terms of eq.(13) with the 2nd order central differencing:

$$u_i^{n+1} = u_i^n - \frac{\nu}{2}(u_{i+1}^n - u_{i-1}^n) + \frac{\nu^2}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \tag{14}$$

where ν is the CFL number expressed as:

$$\nu = \frac{c\Delta t}{\Delta x} \tag{15}$$

The second order derivative term adds numerical dissipation to the scheme and creates the upwind effect. At $\nu = 1$, the Lax-Wendroff scheme returns to the first order upwind scheme which is the accurate solution of the wave equation with no dissipation and dispersion.

The amplification factor of Lax-Wendroff scheme is:

$$G = Re(G) + iIm(G) \tag{16}$$

$$G = 1 - \nu^2(1 - cos\beta) - i\nu sin\beta \tag{17}$$

The magnitude of the amplification factor is defined as:

$$|G| = \sqrt{(Re(G))^2 + (Im(G))^2} \tag{18}$$

and the relative phase error is defined as[8]:

$$\frac{\phi}{\phi_e} = \frac{tan^{-1}[\frac{Im(G)}{Re(G)}]}{-\beta\nu} \tag{19}$$

where $-\beta\nu$ is the exact phase angle of the wave equation, eq.(1). If $\frac{\phi}{\phi_e} = 1$, the numerical scheme has no dispersion error and has the same phase angle as the exact solution of the wave equation. If $\frac{\phi}{\phi_e} > 1$, the numerical scheme is said to have a leading phase error. If $\frac{\phi}{\phi_e} \leq 1$, the numerical scheme has a lagging phase error. The magnitude of the amplification factor and the relative phase error are shown in fig.1 and fig.2, which indicate that the Lax-Wendroff scheme achieves the accurate solution at CFL=1 ($\nu = 1$) with no dissipation and dispersion errors.

However, both the dissipation and dispersion of the Lax-Wendroff schemes increase dramatically when CFL<1. This makes it difficult to control the accuracy when a stretched mesh is used. For a stretched non-uniform mesh, the max CFL number, e.g. CFL=1, is usually set by the smallest mesh cells. Hence the CFL number everywhere else will be less than 1. For unsteady simulations, it then can make the overall solutions very dissipative and dispersive if the mesh is not sufficiently fine. In other words, the high accuracy of the Lax-Wendroff scheme at CFL=1 may be useless if non-uniform mesh is used due to the large variation of dissipation and dispersion with CFL number.

3 Analysis of Runge-Kutta Methods

3.1 2-Stage R-K with 2nd order central differencing

Let $R^{(n)}$ in eq.(7) be evaluated using 2nd order central differencing:

$$R_i^{(n)} = -c\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \tag{20}$$

Substitute it to stage 1 and stage 2 in eq.(5) and (6), then a one step formulation is obtained as:

$$u_i^{n+1} = u_i^n - \frac{\nu}{2}(u_{i+1}^n - u_{i-1}^n) + \frac{\nu^2}{8}(u_{i+2}^n - 2u_i^n + u_{i-2}^n) \tag{21}$$

To facilitate the terminology, the term "stencil" is introduced and defined as:

$$S \equiv \{x_{i1}, x_{i2}, ..., x_{in}\} \tag{22}$$

The width of the stencil is $(in - i1)\Delta x$.

It is obvious that eq.(21) is different from the 2nd order Lax-Wendroff scheme eq.(14). The difference is from the representation of the second order derivative term in the Taylor-series expansion, eq.(13). The 2-stage Runge-Kutta method uses a five point stencil to evaluate the second order derivative while the Lax-Wendroff scheme uses a three point stencil. The five points central differencing does not create the upwind effect as the three points one does in Lax-Wendroff scheme.

The amplification factor of the 2-stage Runge-Kutta scheme with central differencing is:

$$G = 1 - \nu^2 sin^2\beta - i\nu sin\beta \tag{23}$$

$$G^2 = 1 + \frac{\nu^4}{4} sin^4\beta \geq 1 \tag{24}$$

where $\beta = km\Delta x$.

Therefore, the 2-stage Runge-Kutta scheme with 2nd order differencing is unconditionally unstable. This is consistent with the conclusion of Lambert[3][5][9]. The statement in a CFD text book[8](p124) that the standard 2-stage Runge-Kutta method with central differencing is the same as the 2nd order Lax-Wendroff scheme is incorrect.

3.2 2-Stage R-K with 1st Order Alternating One-side Differencing (AOSD)

As mentioned before, both the Lax-Wendroff type schemes and Runge-Kutta schemes are based on the same Taylor-series expansion , eq.(4). Therefore the 2nd order Lax-Wendroff scheme should be able to be obtained by a Runge-Kutta scheme. This is true when the Runge-Kutta scheme is used as a predictor-corrector scheme with alternating direction 1st order one side differencing in each stage, similar to the McCormack's scheme[10].

That is, the $R^{(n)}$ and $R^{(1)}$ in eq.(5) and (6) are evaluated as:

Stage 1, Predictor:

$$R_i^{(n)} = -c\frac{u_{i+1}^n - u_i^n}{\Delta x} \quad (downwind) \tag{25}$$

Stage 2, Corrector:

$$R_i^{(1)} = -c\frac{u_i^{(1)} - u_{i-1}^{(1)}}{\Delta x} \quad (upwind) \tag{26}$$

Then the exact formulation of the Lax-Wendroff scheme, eq.(14), is achieved. That is, the 2nd order Lax-Wendroff scheme can be obtained by 2-stage Runge-Kutta method with alternating one-side 1st order differencing, not by the 2nd order central differencing as erroneously stated in [8]. Using the 2-stage R-K, both the 1st order AOSD and 2nd order central differencing give the 2nd order spatial accuracy. However, the stencil width of the 1st order AOSD is three points and 2nd order central differencing is five points. The 1st order AOSD is stable and the central differencing is unstable.

3.3 4-Stage R-K with 2nd order central differencing

Even though the 2-stage Runge-Kutta method with the 2nd order central differencing is unconditionally unstable, the 4-stage Runge-Kutta method with the 2nd order central differencing is stable with the CFL limit $2\sqrt{2}$, [3][4][5][1]. However, for unsteady calculations, only knowing the stability limit is not enough. The unsteady calculations need to be carried out with minimal dissipation and dispersion errors.

Using the 2nd order central differencing of eq.(20) and substitute it to the 4-stage Runge-Kutta formulations stage by stage from eq.(8) to (11), the following one step formulation is obtained:

$$u_i^{n+1} = u_i^n - \frac{\nu}{2}(u_{i+1}^n - u_{i-1}^n) + \frac{\nu^2}{8}(u_{i+2}^n - 2u_i^n + u_{i-2}^n)$$
$$-\frac{\nu^3}{48}(u_{i+3}^n - 3u_{i+1}^n + 3u_{i-1}^n - u_{i-3}^n)$$
$$+\frac{\nu^4}{384}(u_{i+4}^n - 4u_{i+2}^n + 6u_i^n - 4u_{i-2}^n + u_{i-4}^n) \quad (27)$$

This scheme is 2nd order in space and 4th order in time. Eq.(27) can also be directly obtained by representing the derivatives in the Lax-Wendroff type scheme, eq.(13), with the following central differencing :

$$u_x^n = \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \quad (28)$$

$$u_{xx}^n = \frac{u_{i+2}^n - 2u_i^n + u_{i-2}^n}{4\Delta x^2} \quad (29)$$

$$u_{xxx}^n = \frac{u_{i+3}^n - 3u_{i+1}^n + 3u_{i-1}^n - u_{i-3}^n}{8\Delta x^3} \quad (30)$$

$$u_{xxxx}^n = \frac{u_{i+4}^n - 4u_{i+2}^n + 6u_i^n - 4u_{i-2}^n + u_{i-4}^n}{16\Delta x^4} \quad (31)$$

The first three terms on the right hand side of eq.(27) are exactly the same as those in eq.(21) for the 2-stage Runge-Kutta scheme with central differencing. The 4-stage Runge-Kutta method with the 2nd order central differencing becomes stable due to the added dissipation from the fourth order derivative. From eq.(28) to(31), it can be seen that with the derivative order increased by 1, the stencil width to evaluate the derivative increases by two intervals, one interval on each side.

The amplification factor is:

$$G = 1 + \frac{\nu^2}{4}(cos2\beta - 1) + \frac{\nu^4}{192}(cos4\beta + 3 - 4cos2\beta)$$
$$-i[\frac{\nu^3}{24}(sin3\beta - 3sin\beta) + \nu sin\beta] \quad (32)$$

Fig.3 and Fig.4 are the magnitude of the amplification factor and the relative phase error as defined in eq.(18) and (19). Two conclusions may be drawn from these results:

1) The 4-stage Runge-Kutta scheme with central differencing is essentially dissipation free if CFL$\leq$1. The dissipation free is defined as $0.99 \leq |G| \leq 1.0$. The dissipation free solution is true for both uniform and non-uniform meshes. For the Lax-Wendroff scheme, it can only achieve dissipation free solutions for uniform mesh at CFL=1. This advantage of the Runge-Kutta scheme is very important since a realistic physical problem is usually solved in a non-uniformed mesh. As long as the finest mesh cell has the CFL$\leq$1, all the rest of the mesh cells will have CFL<1.0. Hence the whole flow field will be dissipation free.

The CFL stability limit for this scheme is 2.83. For the CFL between 1 and 2.83, the solution is dissipative and the dissipation increases when the CFL increases.

2) This scheme always has lagging dispersion error, which is independent of CFL number when CFL<2.0. For different CFL numbers, the relative phase error curves coalesce together and reach the maximum at high frequency(e.g. $\beta = \pi$).

3.4 4-Stage R-K with 1st Order AOSD

The same strategy of the predictor-corrector scheme applied to achieve the Lax-Wendroff 2nd order scheme through the 2-stage Runge-Kutta method can also be used for the 4-stage Runge-Kutta scheme. The 1st order one side differencing scheme is used in alternating direction for stage 1 and 2 as in eq.(25) and eq.(26) and repeated in stage 3 and 4. The scheme has the 2nd order accuracy in space and the 4th order accuracy in time.

The one step formulation can be obtained by directly representing the derivatives in the Lax-Wendroff type scheme, eq.(13), with the following central differencing:

$$u_x^n = \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \tag{33}$$

$$u_{xx}^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \tag{34}$$

$$u_{xxx}^n = \frac{u_{i+2}^n - 2u_{i+1}^n + 2u_{i-1}^n - u_{i-2}^n}{2\Delta x^3} \tag{35}$$

$$u_{xxxx}^n = \frac{u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n}{\Delta x^4} \tag{36}$$

The amplification factor is:

$$G = 1 + \nu^2(cos\beta - 1) + \frac{\nu^4}{12}(cos2\beta + 3 - 4cos\beta)$$

$$-i[\frac{\nu^3}{6}(-sin2\beta + 2sin\beta) - \nu sin\beta] \tag{37}$$

Similar to the 1st and 2nd order derivatives in the Lax-Wendroff scheme, the width of the stencil to evaluate the 3rd and 4th order derivatives is the same. The same width of the stencil for the odd and even derivatives creates the upwind effects and increases diffusion. This scheme is hence more dissipative than the one using standard 2nd order central differencing to construct R at each stage, eq.(27). The stability limit also drops from CFL=2.83 to 1.73.

Fig.5 shows the amplification factor of 4-stage Runge-Kutta method with 1st order alternating direction one-side differencing. It is noted that, even though the derivatives from eq.(33) to (36) are represented by central differencing, at no CFL number the scheme is dissipation free. Of course, for steady state solution, the scheme is still dissipation free due to the central differencing. The smaller the CFL number, the smaller the dissipation.

The dispersion also varies largely with CFL number as shown in Fig.6. The optimum CFL number for dispersion is at CFL=1, which does not largely deviate from value 1. At high frequency, unlike the 4-stage R-K with the 2nd order central differencing which has the dispersion curves coalesce together to maximum, the dispersion error of the 1st order AOSD scatters from low to high. This may generate less dispersion error at high frequency than the 2nd order central differencing when a non-uniformed mesh with non-uniform CFL is used.

3.5 4-Stage R-K with 2nd Order Upwind Differencing Scheme

On each stage of the 4-stage R-K method, let $R^{(n)}$ be evaluated by using the 2nd order upwind differencing scheme as:

$$R_i^{(n)} = -c(\frac{\partial u^n}{\partial x})_i = -c(\frac{3u_i^n - 4u_{i-1}^n + u_{i-2}^n}{2\Delta x}) \tag{38}$$

After substituting eq.(38) to the 4-stage R-K from eq.(8) to (11) stage by stage, the one step formulation can be obtained. Due to the lengthy formulations, the one step formulation and the amplification factor are given in Appendix, section 7.1. From the derivative formulations eq.(44) to (47), it can be seen that, with the derivative order increased by 1, the stencil width to evaluate the derivative increases by two intervals, on the upwind side only.

The stability CFL limit for 4-stage R-K with 2nd order upwind differencing is 0.7, far lower than that of the 4-stage R-K with 2nd order central differencing, 2.83.

Fig.7 and Fig.8 are the amplification factor and the relative phase error of the 4-stage R-K with 2nd order upwind differencing. The numerical experiments presented in next section show that all the upwind schemes using with 4-stage R-K have the solutions independent of CFL numbers. With this information in mind, we try to interpret the meaning of fig.7 and fig.8, which is not as obvious as the 2nd order central differencing in fig.3 and 4.

In Fig.7, it is seen that no CFL range can give dissipation free solution. The smaller CFL number has the dissipation varying more monotonically with the frequency. For a large CFL number, the dissipation error increases at mid-range frequency and decreases at high frequency. This may make the dissipation insensitive to CFL number.

In fig.8, the dispersion has mostly leading phase error and scatters for different CFL numbers at high frequency. Similar to the 4-stage with 1st order AOSD, at high frequency, unlike the 4-stage R-K with the 2nd order central differencing with the dispersion coalescing together, the dispersion error of the 2nd order upwind differencing also scatters from low to high. The scattering dispersion error may make the dispersion error insensitive to different CFL numbers. By comparing fig.4 and 8, it can be seen that the dispersion error of the 2nd order upwind differencing is less than the 2nd order central differencing. This is supported by the numerical results in next section.

3.6 4-Stage R-K with 3rd Order Biased Upwind Differencing Scheme

For the 4-stage R-K method, let the $R^{(n)}$ be evaluated by using the 3rd order biased upwind differencing scheme,

$$R_i^{(n)} = -c(\frac{\partial u^n}{\partial x})_i$$
$$= -c(\frac{2u_{i+1}^n + 3u_i^n - 6u_{i-1}^n + u_{i-2}^n}{6\Delta x}) \qquad (39)$$

The one step formulation and amplification factor are given in Appendix, section 7.2. From the derivative formulations eq.(51) to (54), it can be seen that, with the derivative order increased by 1, the stencil width to evaluate the derivative increases by three intervals, one on the downwind side and two on the upwind side.

The CFL stability limit is 1.75, which is twice higher than the 2nd order upwind scheme. Fig.9 and 10 are the amplification factor and relative phase error at different CFL numbers. Similar to the 2nd order upwind scheme, the dissipation exists for all the CFL number range. At high frequency, both the dissipation and dispersion curves scatter with different CFL numbers. This makes the dissipation and dispersion error insensitive to CFL number. Unlike the 2nd order upwind differencing which mostly has the leading phase error, the dispersion of the 3rd order upwind scheme is the lagging phase error.

3.7 4-Stage R-K with 4th Order Central Differencing Scheme

For the 4-stage R-K method, let the $R^{(n)}$ be evaluated by using the 4th order central differencing scheme,

$$R_i^{(n)} = -c(\frac{\partial u^n}{\partial x})_i$$
$$= -c(\frac{-u_{i+2}^n + 8u_{i+1}^n - 8u_{i-1}^n + u_{i-2}^n}{12\Delta x}) \qquad (40)$$

The one step formulation and amplification factor are given in Appendix, section 7.3. From the derivative formulations eq.(57) to (60), it can be seen that, with the derivative order increased by 1, the stencil width to evaluate the derivative increases by four intervals, two on each side.

The CFL stability limit is 2.06, which is lower than that of the 2nd order central differencing, 2.83. Fig.11 is the amplification factor at different CFL numbers. The dissipation pattern is very similar to the 2nd order central differencing. The scheme is essentially dissipation free when CFL$\leq$0.8.

Fig.12 is the relative phase error at different CFL numbers. The dispersion error is essentially independent of CFL number when CFL$\leq$1.5. The overall dispersion error of the 4th order central differencing is significantly less than that of the 2nd order differencing. This is shown as the dispersion error curve of the 2nd order central differencing (fig.4) is steeper than that of the 4th order central differencing. Like the 2nd order central differencing, the relative phase error curves coalesce together and reach the maximum at high frequency for all the CFL numbers.

3.8 4-Stage R-K with 4th Order Biased Upwind Differencing Scheme

For the 4-stage R-K method, let the $R^{(n)}$ be evaluated by using the 4th order biased upwind differencing scheme,

$$R_i^{(n)} = -c\frac{\partial u^n}{\partial x}_i$$
$$= -c(\frac{3u_{i+1}^n + 10u_i^n - 18u_{i-1}^n + 6u_{i-2}^n - u_{i-3}^n}{12\Delta x}) \qquad (41)$$

The one step formulation and amplification factor are given in Appendix, section 7.4. From the derivative formulations eq.(63) to (66), it can be seen that, with the derivative order increased by 1, the stencil width to evaluate the derivative increases by four intervals, one on the downwind side and three on upwind side.

The CFL stability limit is 1.05, which is lower than that of the 4th order central differencing.

Fig.13 and 14 are the amplification factor and the relative phase error at different CFL numbers. Similar to all the other upwind schemes, the dissipation exists for all the CFL number range. The dissipation and dispersion is less than the 2nd order upwind and 3rd order biased upwind scheme, and also scatter at high frequency.

3.9 Summary of the Analytical Results

Table 1 (on the last page of this paper) summarizes the performance of each schemes studied. The table lists the CFL stability limit, dissipation free CFL limit, and the dispersion independent CFL limit. The stability limit is determined when the magnitude of the amplification factor is slightly greater than 1.0 at any frequency point. In general, the 4-stage Runge-Kutta method using central differencing schemes has higher stability than using upwind differencing schemes with the same oder of accuracy. The highest CFL stability limit is the 4-stage R-K with 2nd order central differencing, CFL=2.83. In addition, the 4-stage R-K with central differencing has a CFL range within which the solution is essentially dissipation free, while the upwind schemes are dissipative for all the CFL number range.

All the schemes have dispersion error, which reaches the maximum at high frequency for central differencing. However, the central differencing schemes have a CFL range within which the dispersion error is independent of the CFL number. The dispersion independent CFL limit is higher than the dissipation free CFL limit. Hence, if the central differencing scheme is within the dissipation free range, it is also in the dispersion independent CFL range. The dissipation and dispersion of the upwind schemes are insensitive to the CFL numbers. The numerical results presented in the next section support this conclusion. This is important to ensure that the whole flow field has about the same level of dissipation and dispersion when the non-uniform mesh is used.

4 Numerical Results

4.1 Wave Equation Solution

The wave equation, eq.(1), with $c = 1$ and wave number $n = 3$ is solved. This may represent a low frequency wave solution. The initial solution is:

$$u(x,0) = sin2n\pi(\frac{x}{40}), \qquad 0 \le x \le 40. \quad (42)$$

The analytical solution with periodic boundary conditions is:

$$u(x,t) = sin2n\pi(\frac{x-t}{40}), \qquad 0 \le x \le 40. \quad (43)$$

4.2 Results for All the Schemes

The number of mesh points used is 81. For all the schemes studied, the numerical CFL stability limits agree with the analytical ones given in table 1. That is, when the CFL number is equal or slightly greater than the analytical CFL stability limits, the solutions diverge.

Fig.15 shows the results of the Lax-Wendroff scheme and the 4-Stage Runge-Kutta method with 2nd order upwind differencing at time level t=720. At CFL=1, the Lax-Wendroff scheme precisely reproduces the analytical solution because the scheme is dissipation and dispersion free at CFL=1. However, once the CFL is slightly away from 1.0, the solution is seriously diffused. Fig.15 shows that the result at CFL=0.98 has the peak value only about 1/3 of the analytical solution, and the result at CFL=0.9 is completely smeared to zero.

As mentioned before, if the Lax-Wendroff scheme is used for a non-uniform mesh, only the smallest grid points can have CFL=1 and all other grid points will have the CFL less than 1. Therefore, the unsteady solution could be seriously diffused unless extremely fine mesh is used. However, the severe temporal dissipation does not prevent the scheme to be used for steady state calculation such as the Ni's scheme[11][12], because the accuracy of the steady state solution is controlled by the spatial differencing scheme. McCormack's scheme is also equivalent to the Lax-Wendroff scheme[10].

Fig.15 shows that the results of the 4-Stage Runge-Kutta method with 2nd order upwind differencing is also seriously diffused as expected, but it is less diffusive than the Lax-Wendroff scheme even though both schemes have the 2nd order accuracy in space. It is observed that the solutions of 4-Stage Runge-Kutta method with 2nd order upwind differencing is not sensitive to the CFL number. Both the solutions of the CFL=0.68 and CFL=0.068 collapse together.

Fig.16 is the results of the 4-Stage Runge-Kutta method with the 1st order AOSD. Similar to the Lax-Wendroff scheme, the dissipation varies with the CFL. The smaller CFL, the less the dissipation as indicated in fig.5. The phase error also varies with CFL number and is very large. Compared with the 4-Stage Runge-Kutta method with the 2nd order upwind differencing in fig.15, the AOSD is less diffusive, but the phase error is larger.

Fig.17 shows the numerical results of the 4-stage Runge-Kutta method with 2nd and 4th order central differencing at time level 720. The results of the 2nd order central differencing with CFL=1.0

and CFL=0.1 have the same peak value as the analytical solution because the CFL numbers are within the dissipation free range, CFL$\leq$1.0. The two solutions also collapse together with the same phase shift from the analytical solution. This is because the CFL numbers are in the dispersion independent range, CFL$\leq$2.0. The dispersion error is so large that the phase of the numerical solutions are shifted by about half a cycle compared to the analytical solution. Even though the wave shape and peak value are well preserved, the result is completely wrong with such a large phase shift. Comparing fig.17 and 15, it can be seen that the 2nd order upwind scheme has less dispersion error than the 2nd order central differencing.

Fig.17 also shows the result of the 4-stage Runge-Kutta method with 2nd order central differencing at CFL=2.5, which is outside of the dissipation free and dispersion independent range. The solution is smeared and the phase shift is different from the one with CFL=1.0.

In fig.17, the result of the 4-stage Runge-Kutta method with 4th order central differencing at CFL=1 agrees very well with the analytical solution. This is consistent with the analytical result that the 4-stage R-K with 4th order central differencing has less dispersion error than the 2nd order central differencing within the dispersion independent CFL range.

If the time level is increased by 10 times to t=7200, the 4th order central differencing result with CFL=0.8 shows a large phase error as presented in fig.18. The result preserves the peak value accurately since it is in the dissipation free range, CFL$\leq$0.8. The solution of the 4th order central differencing with CFL=2.0 shows large dissipation and dispersion error, and the result of the 2nd order central differencing with CFL=2.8 is the worst. This is because both of the CFL numbers are outside of their dissipation and dispersion independent range.

The results on fig.18 indicate that, for the unsteady calculation of the wave equation using 4-stage R-K with a central differencing, it could be very diffusive if only limiting the CFL in the stability range. The CFL should be limited in the dissipation free range.

Fig.19 shows the numerical results of the 4-stage Runge-Kutta method with 3rd and 4th order biased upwind differencing at time level 720. The results of the 3rd order biased upwind differencing with CFL=1.74, 0.87, and 0.087 all collapse together and the solutions are seriously diffused with a phase error. This indicates that the dissipation and dispersion of the 4-stage R-K with 3rd upwind differencing are independent of the CFL number in the CFL stability range. Similar results are also shown for the 4-stage R-K with 2nd upwind differencing in fig.15. At the time level 720, the result of the 4th order biased upwind differencing agrees very well with the analytical one, which proves that the 4th order biased upwind differencing has less dissipation and dispersion error than the 3rd order biased upwind differencing.

Again, if the time level is increased by 10 times higher to t=7200 as shown in fig.20, the accuracy of the 4th order biased upwind differencing is lost due to its inherent dissipation and dispersion error. Fig.20 shows that the results of the 4-stage R-K with 4th order biased upwind differencing collapse together for CFL=1.0, 0.2, and 0.02 with a dissipation and dispersion error. This indicates that the dissipation and dispersion of the 4-stage R-K with 4th order upwind differencing are independent of the CFL number in the CFL stability range. Fig.20 also presents the results of the 4-stage R-K with 3rd order biased upwind differencing at CFL=1.0 and 0.2. The results are almost smeared to being flat. Again, the results collapse together as in fig.19. Fig.20 means that the dissipation and dispersion error of the 4th order biased upwind scheme is significantly less than those of the 3rd order biased upwind differencing.

4.3 Compare the 4th Order Schemes

The 4-Stage R-K has the 4th order accuracy in temporal direction. If a spatial 4th order accuracy scheme is used, then a complete 4th order accuracy scheme in space and time is created. It is hence of interest to compare the two spatial 4th order schemes: the 4th order central differencing and the 4th order biased upwind scheme.

First of all, if CFL$\leq$0.8, the 4th order central differencing will only have dispersion error and no dissipation error. The 4th order biased upwind scheme will have both dissipation and dispersion error at all CFL range. By comparing the relative phase error of these two schemes in fig.12 and 14, it is seen that the low dispersion error area (around value 1) of the 4th order biased upwind scheme is larger than that of the 4th order central differencing. At the high frequency range, the dispersion of the biased upwind scheme scatters while that of the central differencing converges to the maximum error. Hence, the 4th order biased upwind scheme has less dispersion error than the 4th order central differencing.

Fig.17 and 19 show that both the 4th order central differencing and 4th order biased upwind differencing (the circle symbols) agree very well with the analytical solution at time level t=700. When the time level is increased to t=7000 as shown in fig.18 and 20, both do not agree well with the analytical solution. The central differencing results has a lagging phase shift and the upwind differencing has both a leading phase shift and peak value smeared. When the mesh size is doubled to 161, both agree excellently with the analytical solution at t=7200 as shown in fig.21. This is because all these schemes are consistent and the dissipation and dispersion error can always be reduced by using finer mesh.

Above results seem indicating that the 4th order central differencing and 4th order biased upwind differencing have similar level of accuracy for the low frequency wave equation solutions studied in this paper. That is, when one scheme is accurate or inaccurate, so is the other one even though the nature of the error is different. Of course, the accuracy is a relative measurement. With the time level further increased beyond t=7200, the dissipation and dispersion error will also increase even with the refined mesh. For a particular physical problem, we may not have to find an absolute dissipation and dispersion free scheme. What we need to find is the scheme and mesh size that are sufficiently accurate to the required time level, such as the results in fig.21. It is hoped that the work of this paper can give some help to serve this purpose.

5 Conclusion

The von Neumann analysis is carried out to study the dissipation, dispersion and stability limits of the unsteady linear wave equation solved by the standard 4-stage Runge-Kutta method with several widely used spacial differencing schemes, including 2nd order central differencing, 2nd order upwind, 3rd order and 4th order biased upwind, and 4th order central differencing. The 2nd order Lax-Wendroff scheme and the 2-stage Runge-Kutta method are also analyzed as references.

The 2-stage Runge-Kutta method with the 2nd order central differencing is unconditionally unstable. The statement in [8] that the 2-stage Runge-Kutta method with the 2nd order central differencing is the same as the 2nd order Lax-Wendroff scheme is incorrect. The 2nd order Lax-Wendroff scheme can only be achieved by using the 1st order alternating one-side differencing with the 2-stage Runge-Kutta method.

The Lax-Wendroff scheme is accurate for the linear wave equation at CFL=1.0. However, when CFL is off 1.0, the scheme is extremely dissipative. This may make the Lax-Wendroff scheme too diffusive to be used for unsteady calculation on a non-uniform mesh, where only the smallest mesh points can have CFL=1, and everywhere else will have CFL$<$1.

The 1st order alternating one-side differencing can also be used with 4-stage Runge-Kutta method, which has the 2nd order accuracy of central differencing in space and 4th order accuracy in time, and the stability limit is CFL$\leq$1.73. At no CFL number, the scheme is dissipation free.

For a central differencing with the 4-stage Runge-Kutta method, there is a CFL limit, under which the solution is dissipation free. The dissipation free CFL limit is far below the stability CFL limit. There is also a CFL limit under which the dispersion of the central differencing is independent of the CFL number. The dispersion independent CFL limit is higher than the dissipation free CFL limit.

For the 2nd order central differencing with the 4-stage Runge-Kutta method, the stability limit is CFL$\leq$2.83, the dissipation free limit is CFL$\leq$1.0, and the dispersion independent CFL limit is CFL$\leq$2.0. For the 4th order central differencing with the 4-stage Runge-Kutta method, the stability limit is CFL$\leq$2.06, the dissipation free limit is CFL$\leq$0.8, and the dispersion independent CFL limit is CFL$\leq$1.5.

The numerical results indicated that the dissipation and dispersion error of upwind schemes with 4-stage Runge-Kutta method are independent of the CFL number under the CFL stability limit. The CFL stability limits of the 2nd order upwind differencing, 3rd order and 4th order biased upwind differencing are 0.7, 1.75, and 1.05 respectively.

Overall, an upwind scheme with the 4-stage Runge-Kutta method has lower stability limit and dispersion error than that of the central differencing with the same order of accuracy. The dispersion error exists for all the schemes with the 4-stage Runge-Kutta method.

For the wave equation with a low frequency solution studied in this paper, the 4th order central differencing and the 4th order biased upwind differencing have similar level of accuracy.

6 Acknowledgment

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7 Appendix

7.1 4-Stage R-K with 2nd Order Upwind Differencing Scheme

The one step formulation can be obtained by directly representing the derivatives in the Lax-Wendroff type scheme, eq.(13), with the following upwind differencing :

$$u_x^n = \frac{3u_i^n - 4u_{i-1}^n + u_{i-2}^n}{2\Delta x} \tag{44}$$

$$u_{xx}^n = \frac{9u_i^n - 24u_{i-1}^n + 22u_{i-2}^n - 8u_{i-3}^n + u_{i-4}^n}{4\Delta x^2} \tag{45}$$

$$u_{xxx}^n = \frac{27u_i^n - 108u_{i-1}^n + 171u_{i-2}^n - 136u_{i-3}^n}{8\Delta x^3} + \frac{57u_{i-4}^n - 12u_{i-5}^n + u_{i-6}^n}{8\Delta x^3} \tag{46}$$

$$u_{xxxx}^n = \frac{81u_i^n - 432u_{i-1}^n + 972u_{i-2}^n - 1200u_{i-3}^n}{16\Delta x^4} + \frac{886u_{i-4}^n - 400u_{i-5}^n + 108u_{i-6}^n}{16\Delta x^4} + \frac{-16u_{i-7}^n + u_{i-8}^n}{16\Delta x^4} \tag{47}$$

The amplification factor is:

$$G = Re(G) + iIm(G) \tag{48}$$

where

$$Re(G) = 1 - \frac{\nu}{2}(3 - 4cos\beta + cos2\beta)$$

$$+\frac{\nu^2}{8}(9 - 24cos\beta + 22cos2\beta - 8cos3\beta + cos4\beta)$$

$$-\frac{\nu^3}{48}(27 - 108cos\beta + 171cos2\beta - 136cos3\beta + 57cos4\beta - 12cos5\beta + cos6\beta)$$

$$+\frac{\nu^4}{384}(81 - 432cos\beta + 972cos2\beta - 1200cos3\beta + 886cos4\beta - 400cos5\beta + 108cos6\beta - 16cos7\beta + cos8\beta) \tag{49}$$

$$Im(G) = -\frac{\nu}{2}(4sin\beta - sin2\beta)$$

$$+\frac{\nu^2}{8}(24sin\beta - 22sin2\beta + 8sin3\beta - sin4\beta)$$

$$-\frac{\nu^3}{48}(108sin\beta - 171sin2\beta + 136sin3\beta - 57sin4\beta + 12sin5\beta - sin6\beta)$$

$$+\frac{\nu^4}{384}(432sin\beta - 972sin2\beta + 1200sin3\beta - 886sin4\beta + 400sin5\beta - 108sin6\beta + 16sin7\beta - sin8\beta) \tag{50}$$

7.2 4-Stage R-K with 3rd Order Biased Upwind Differencing Scheme

The one step formulation can be obtained by directly representing the derivatives in the Lax-Wendroff type scheme, eq.(13), with the following biased upwind differencing :

$$u_x^n = \frac{2u_{i+1}^n + 3u_i^n - 6u_{i-1}^n + u_{i-2}^n}{6\Delta x} \tag{51}$$

$$u_{xx} = \frac{4u_{i+2}^n + 12u_{i+1}^n - 15u_i^n - 32u_{i-1}^n}{36\Delta x^2} + \frac{42u_{i-2}^n - 12u_{i-3}^n + u_{i-4}^n}{36\Delta x^2} \tag{52}$$

$$u_{xxx}^n = \frac{8u_{i+3}^n + 36u_{i+2}^n - 18u_{i+1}^n - 177u_i^n}{216\Delta x^3} + \frac{90u_{i-1}^n + 279u_{i-2}^n - 318u_{i-3}^n + 117u_{i-4}^n}{216\Delta x^3} + \frac{-18u_{i-5}^n + u_{i-6}^n}{216\Delta x^3} \tag{53}$$

$$u_{xxxx}^n = \frac{16u_{i+4}^n + 96u_{i+3}^n + 24u_{i+2}^n - 616u_{i+1}^n}{1296\Delta x^4} + \frac{-207u_i^n + 1872u_{i-1}^n - 516u_{i-2}^n - 2304u_{i-3}^n}{1296\Delta x^4} + \frac{2502u_{i-4}^n - 1072u_{i-5}^n + 228u_{i-6}^n}{1296\Delta x^4} + \frac{-24u_{i-7}^n + u_{i-8}^n}{1296\Delta x^4} \tag{54}$$

The amplification factor is in the form of eq.(48):

where

$$Re(G) = 1 - \frac{\nu}{6}(3 - 4cos\beta + cos2\beta)$$

$$+\frac{\nu^2}{72}(-15 - 20cos\beta + 46cos2\beta - 12cos3\beta + cos4\beta)$$

$$-\frac{\nu^3}{1296}(-177 + 72cos\beta + 315cos2\beta - 310cos3\beta$$

$$+117cos4\beta - 18cos5\beta + cos6\beta)$$

$$+\frac{\nu^4}{31104}(-207 + 1256cos\beta - 492cos2\beta - 2208cos3\beta$$

$$+2518cos4\beta - 1072cos5\beta + 228cos6\beta$$

$$-24cos7\beta + cos8\beta) \quad (55)$$

$$Im(G) = -\frac{\nu}{6}(8sin\beta - sin2\beta)$$

$$+\frac{\nu^2}{72}(44sin\beta - 38sin2\beta + 12sin3\beta - sin4\beta)$$

$$-\frac{\nu^3}{1296}(-108sin\beta - 243sin2\beta + 326sin3\beta$$

$$-117sin4\beta + 18sin5\beta - sin6\beta)$$

$$+\frac{\nu^4}{31104}(-2488sin\beta + 540sin2\beta + 2400sin3\beta$$

$$-2486sin4\beta + 1072sin5\beta - 228sin6\beta$$

$$+24sin7\beta - sin8\beta) \quad (56)$$

7.3 4-Stage R-K with 4th Order Central Differencing

The one step formulation can be obtained by directly representing the derivatives in the Lax-Wendroff type scheme, eq.(13), with the following central differencing :

$$u_x^n = \frac{-u_{i+2}^n + 8u_{i+1}^n - 8u_{i-1}^n + u_{i-2}^n}{12\Delta x} \quad (57)$$

$$u_{xx}^n = \frac{u_{i+4}^n - 16u_{i+3}^n + 64u_{i+2}^n + 16u_{i+1}^n}{144\Delta x^2}$$

$$+\frac{-130u_i^n + 16u_{i-1}^n + 64u_{i-2}^n - 16u_{i-3}^n + u_{i-4}^n}{144\Delta x^2} \quad (58)$$

$$u_{xxx}^n = \frac{-u_{i+6}^n + 24u_{i+5}^n - 192u_{i+4}^n + 488u_{i+3}^n}{1728\Delta x^3}$$

$$+\frac{387u_{i+2}^n - 1584u_{i+1}^n + 1584u_{i-1}^n - 387u_{i-2}^n}{1728\Delta x^3}$$

$$+\frac{-488u_{i-3}^n + 192u_{i-4}^n - 24u_{i-5}^n + u_{i-6}^n}{1728\Delta x^3} \quad (59)$$

$$u_{xxxx}^n = \frac{u_{i+8}^n - 32u_{i+7}^n + 384u_{i+6}^n - 2016u_{i+5}^n}{20736\Delta x^4}$$

$$+\frac{3324u_{i+4}^n + 6240u_{i+3}^n - 16768u_{i+2}^n - 4192u_{i+1}^n}{20736\Delta x^4}$$

$$+\frac{26118u_i^n - 4192u_{i-1}^n - 16768u_{i-2}^n + 6240u_{i-3}^n}{20736\Delta x^4}$$

$$+\frac{3324u_{i-4}^n - 2016u_{i-5}^n}{20736\Delta x^4}$$

$$\frac{+384u_{i-6}^n - 32u_{i-7}^n + u_{i-8}^n}{20736\Delta x^4} \quad (60)$$

The amplification factor is in the form of eq.(48):

where

$$Re(G) = 1 + \frac{\nu^2}{288}(-130 + 32cos\beta + 128cos2\beta$$

$$-32cos3\beta + 2cos4\beta) + \frac{\nu^4}{497664}(26118 - 8384cos\beta$$

$$-33536cos2\beta + 12480cos3\beta + 6648cos4\beta$$

$$-4032cos5\beta + 768cos6\beta - 64cos7\beta + 2cos8\beta) \quad (61)$$

$$Im(G) = \frac{\nu}{12}(-16sin\beta + 2sin2\beta)$$

$$+\frac{\nu^3}{10368}(3168sin\beta - 774sin2\beta - 976sin3\beta$$

$$+384sin4\beta - 48sin5\beta + 2sin6\beta) \quad (62)$$

7.4 4-Stage R-K with 4th Order Biased Upwind Differencing Scheme

The one step formulation can be obtained by directly representing the derivatives in the Lax-Wendroff type scheme, eq.(13), with the following biased upwind differencing :

$$u_x^n = \frac{3u_{i+1}^n + 10u_i^n - 18u_{i-1}^n + 6u_{i-2}^n - u_{i-3}^n}{12\Delta x} \quad (63)$$

$$u_{xx}^n = \frac{9u_{i+2}^n + 60u_{i+1}^n - 8u_i^n - 324u_{i-1}^n + 438u_{i-2}^n}{144\Delta x^2}$$

$$+\frac{-236u_{i-3}^n + 72u_{i-4}^n - 12u_{i-5}^n + u_{i-6}^n}{144\Delta x^2} \quad (64)$$

$$u_{xxx}^n = \frac{27u_{i+3}^n + 270u_{i+2}^n + 414u_{i+1}^n - 2078u_i^n}{1728\Delta x^3}$$

$$+\frac{-1431u_{i-1}^n + 9396u_{i-2}^n - 11964u_{i-3}^n + 7884u_{i-4}^n}{1728\Delta x^3}$$

$$+\frac{-3267u_{i-5}^n + 894u_{i-6}^n - 162u_{i-7}^n + 18u_{i-8}^n - u_{i-9}^n}{1728\Delta x^3} \quad (65)$$

$$u_{xxxx}^n = \frac{81u_{i+4}^n + 1080u_{i+3}^n + 3456u_{i+2}^n - 6792u_{i+1}^n}{20736\Delta x^4}$$

$$+\frac{-30932u_i^n + 53496u_{i-1}^n + 70944u_{i-2}^n}{20736\Delta x^4}$$

$$+\frac{-271624u_{i-3}^n + 342198u_{i-4}^n - 253080u_{i-5}^n}{20736\Delta x^4}$$

$$+\frac{126528u_{i-6}^n - 45144u_{i-7}^n + 11724u_{i-8}^n}{20736\Delta x^4}$$

$$+\frac{-2200u_{i-9}^n + 288u_{i-10}^n - 24u_{i-11}^n + u_{i-12}^n}{20736\Delta x^4} \quad (66)$$

The amplification factor is in the form of eq.(48):

where

$$Re(G) = 1 - \frac{\nu}{12}(10 - 15cos\beta + 6cos2\beta - cos3\beta)$$
$$+\frac{\nu^2}{288}(-8 - 264cos\beta + 447cos2\beta - 236cos3\beta$$
$$+72cos4\beta - 12cos5\beta + cos6\beta)$$
$$-\frac{\nu^3}{10368}(-2078 - 1017cos\beta + 9666cos2\beta$$
$$-11937cos3\beta + 7884cos4\beta - 3267cos5\beta$$
$$+894cos6\beta - 162cos7\beta + 18cos8\beta - cos9\beta)$$
$$+\frac{\nu^4}{497664}(-30932 + 46704cos\beta + 74400cos2\beta$$
$$-270544cos3\beta + 342279cos4\beta - 253080cos5\beta$$
$$+126528cos6\beta - 45144cos7\beta + 11724cos8\beta$$
$$-2200cos9\beta + 288cos10\beta - 24cos11\beta + cos12\beta) \quad (67)$$

$$Im(G) = -\frac{\nu}{12}(21sin\beta - 6sin2\beta + sin3\beta)$$
$$+\frac{\nu^2}{288}(384sin\beta - 429sin2\beta + 236sin3\beta$$
$$-72sin4\beta + 12sin5\beta - sin6\beta)$$
$$-\frac{\nu^3}{10368}(1845sin\beta - 9126sin2\beta + 11991sin3\beta$$
$$-7884sin4\beta + 3267sin5\beta - 894sin6\beta$$
$$+162sin7\beta - 18sin8\beta + sin9\beta)$$
$$+\frac{\nu^4}{497664}(-60288sin\beta - 67488sin2\beta + 272704sin3\beta$$
$$-342117sin4\beta + 253080sin5\beta - 126528sin6\beta$$
$$+45144sin7\beta - 11724sin8\beta + 2200sin9\beta$$
$$-288sin10\beta + 24sin11\beta - sin12\beta) \quad (68)$$

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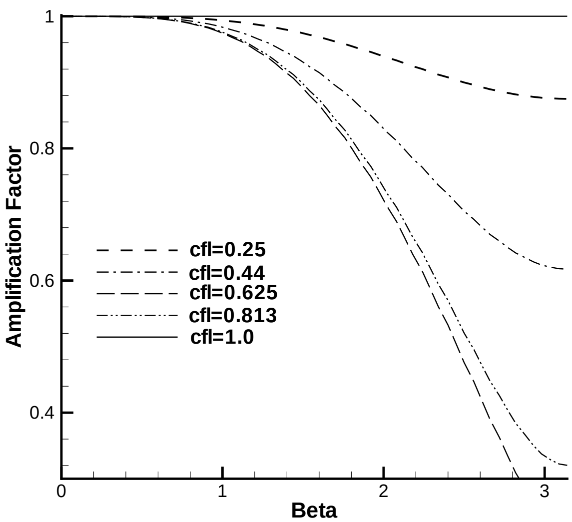

Figure 1: Amplification factor of the 2nd order Lax-Wendroff scheme

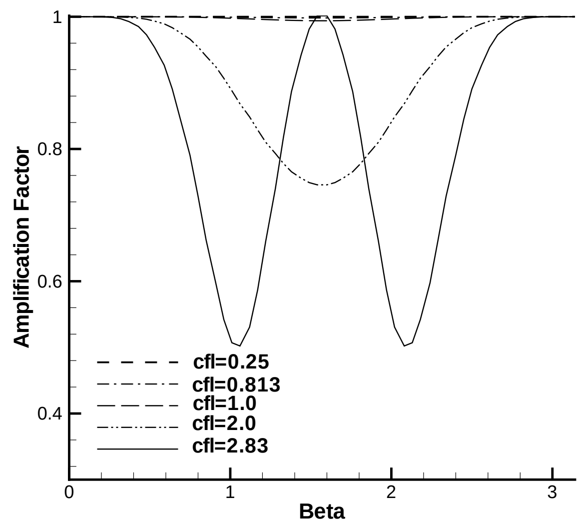

Figure 3: Amplification factor of 4-Stage Runge-Kutta method with 2nd order central differencing

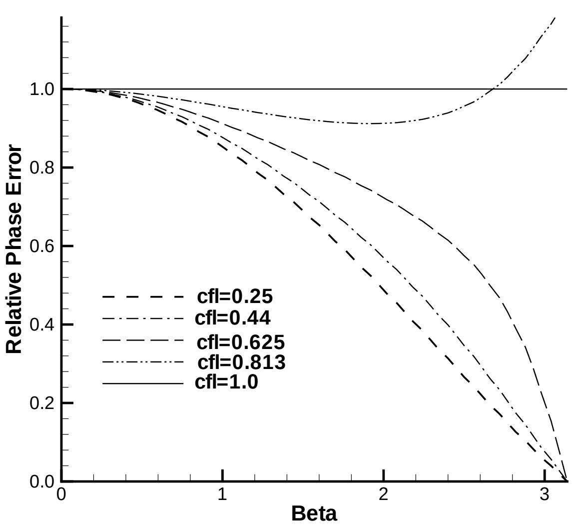

Figure 2: Relative phase error of the 2nd order Lax-Wendroff scheme

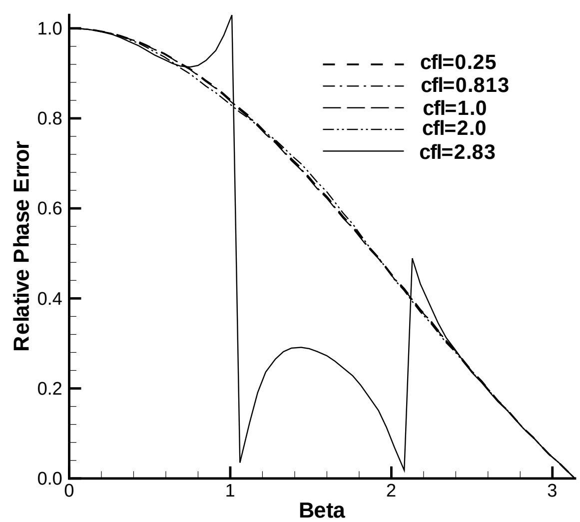

Figure 4: Relative phase error of 4-Stage Runge-Kutta method with 2nd order central differencing

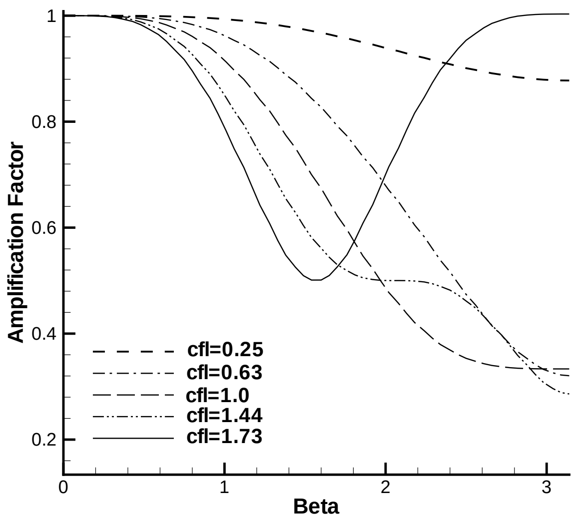

Figure 5: Amplification factor of 4-Stage Runge-Kutta method with the 1st order alternating one side differencing scheme

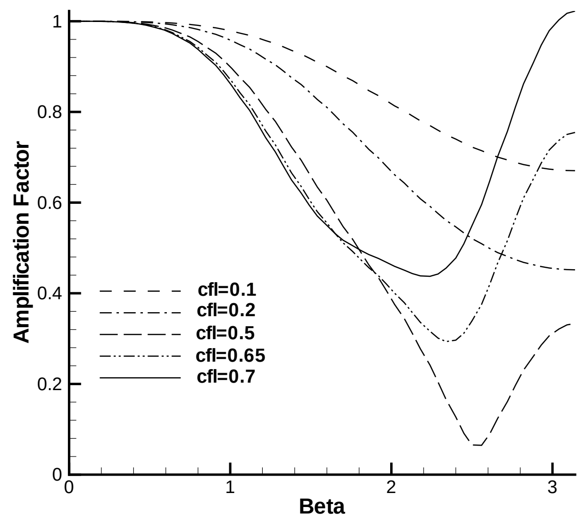

Figure 7: Amplification factor of 4-Stage Runge-Kutta method with 2nd order upwind differencing scheme

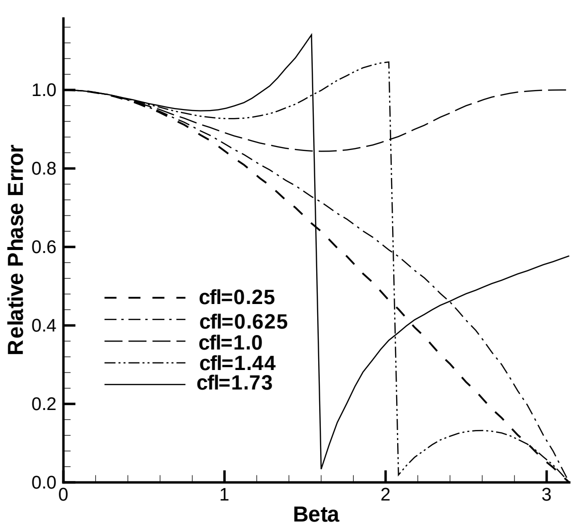

Figure 6: Relative phase error of 4-Stage Runge-Kutta method with the 1st order alternating one side differencing scheme

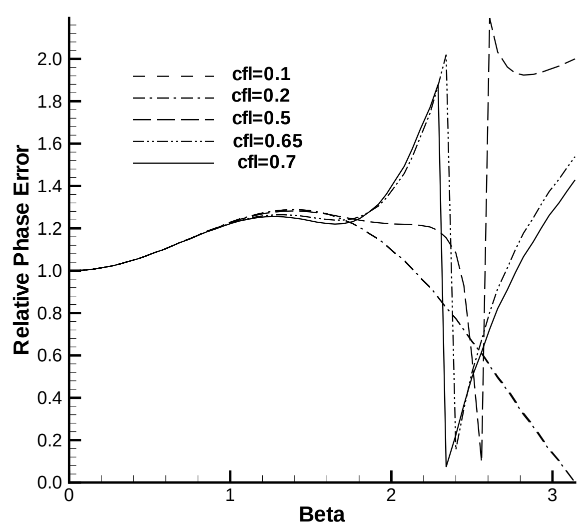

Figure 8: Relative phase error of 4-Stage Runge-Kutta method with 2nd order upwind differencing scheme

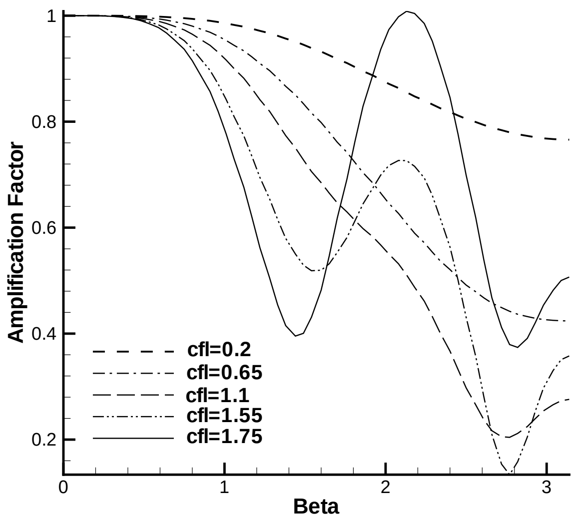

Figure 9: Amplification factor of 4-Stage Runge-Kutta method with 3rd order biased upwind differencing scheme

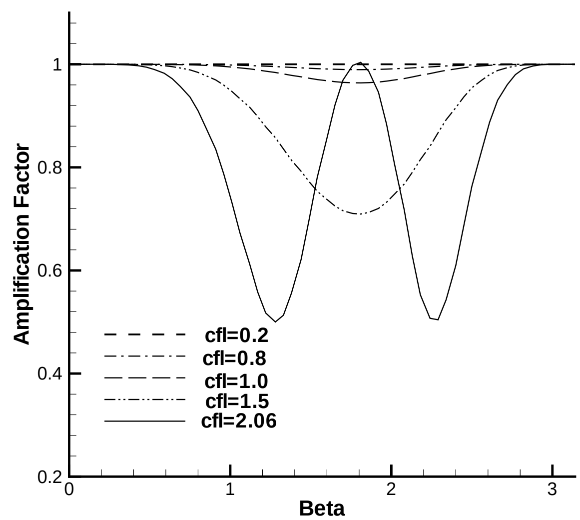

Figure 11: Amplification factor of 4-Stage Runge-Kutta method with 4th order central differencing

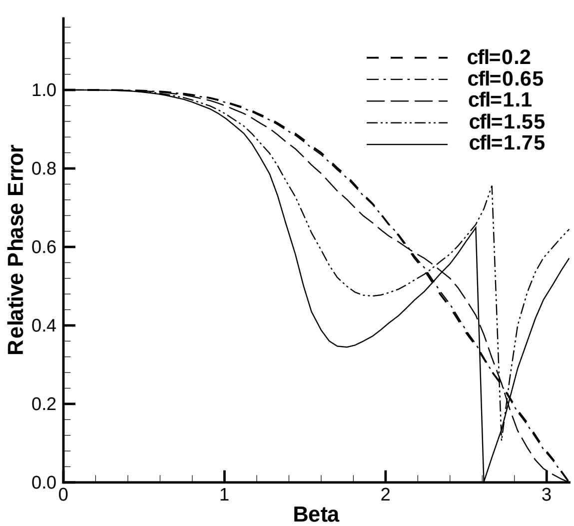

Figure 10: Relative phase error of 4-Stage Runge-Kutta method with 3rd order biased upwind differencing Scheme

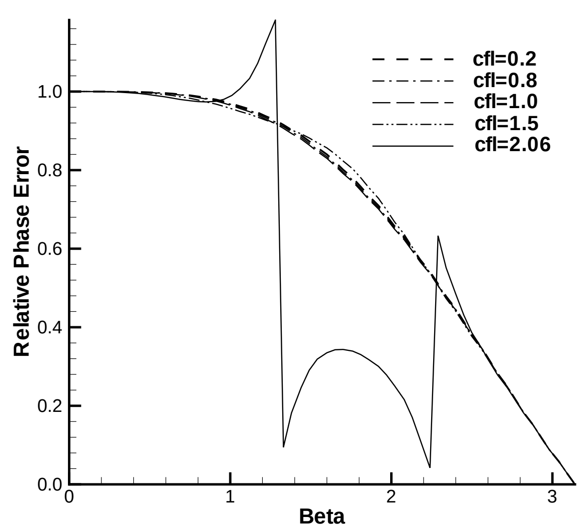

Figure 12: Relative phase error of 4-Stage Runge-Kutta method with 4th order central differencing

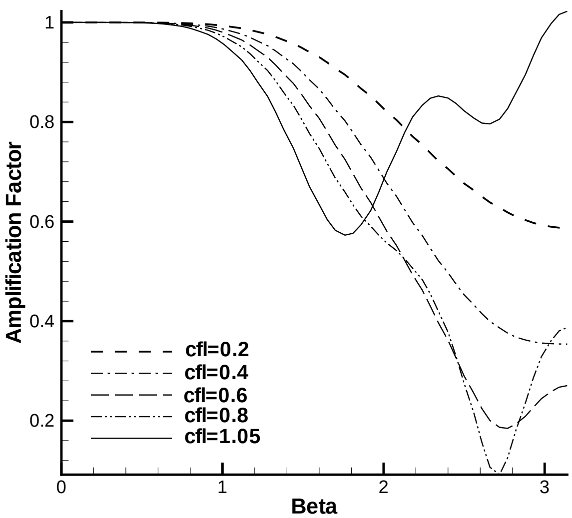

Figure 13: Amplification factor of 4-Stage Runge-Kutta method with 4th order biased upwind differencing scheme

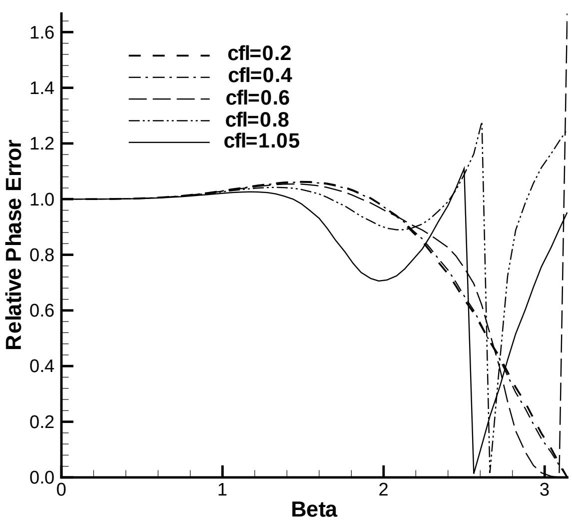

Figure 14: Relative phase error of 4-Stage Runge-Kutta method with 4th order biased upwind differencing scheme

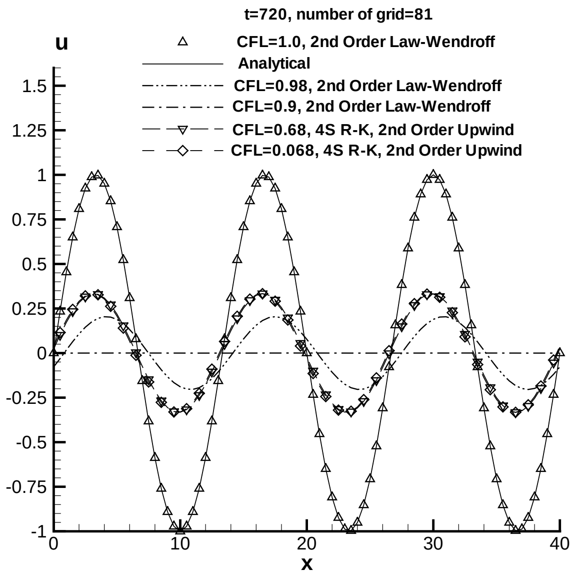

Figure 15: Numerical solutions of the wave equation for Lax-Wendroff scheme and 4-Stage Runge-Kutta method with 2nd order upwind differencing, t=720

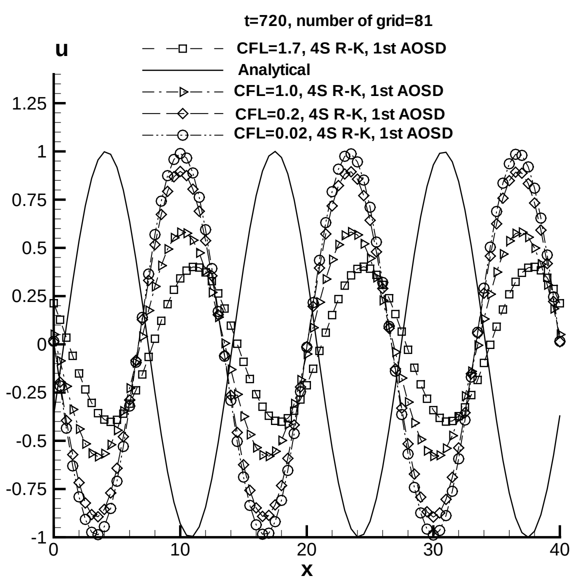

Figure 16: Numerical solutions of the wave equation for 4-Stage Runge-Kutta method with 1st order AOSD, t=720

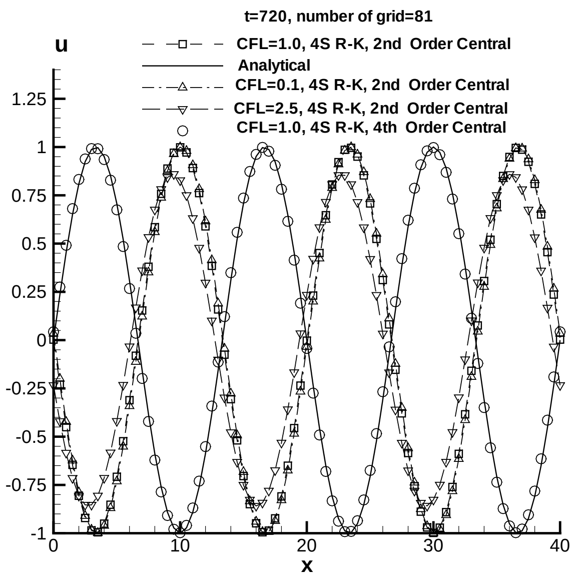

Figure 17: Numerical solutions of the wave equation for 4-Stage Runge-Kutta method with 2nd and 4th order central differencing, t=720

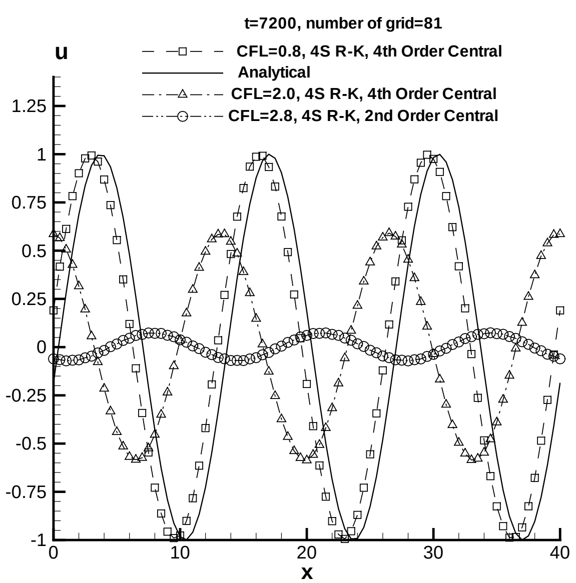

Figure 18: Numerical solutions of the wave equation for 4-Stage Runge-Kutta method with 2nd and 4th order order central differencing, t=7200.

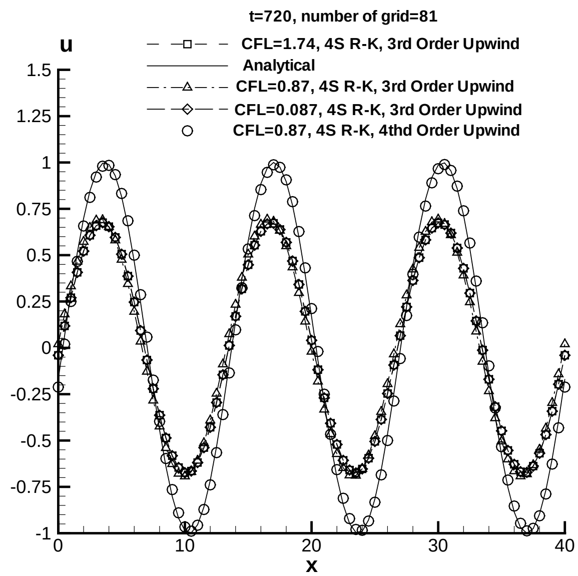

Figure 19: Numerical solutions of the wave equation for 4-Stage Runge-Kutta method with 3rd and 4th order biased Upwind differencing, t=720.

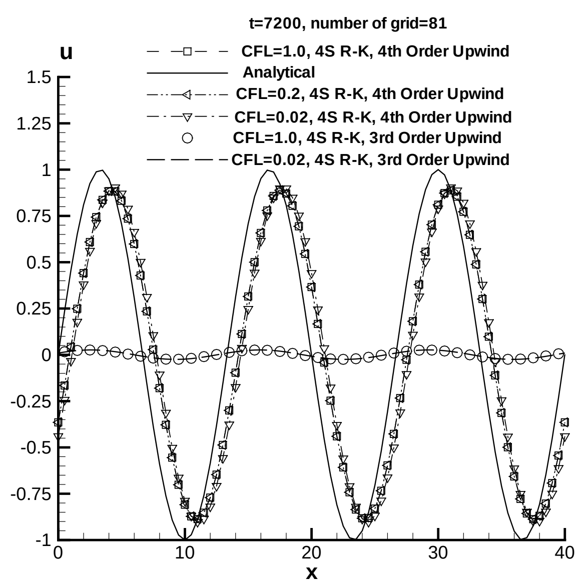

Figure 20: Numerical solutions of the wave equation for 4-Stage Runge-Kutta method with 3rd and 4th order biased Upwind differencing, t=720.

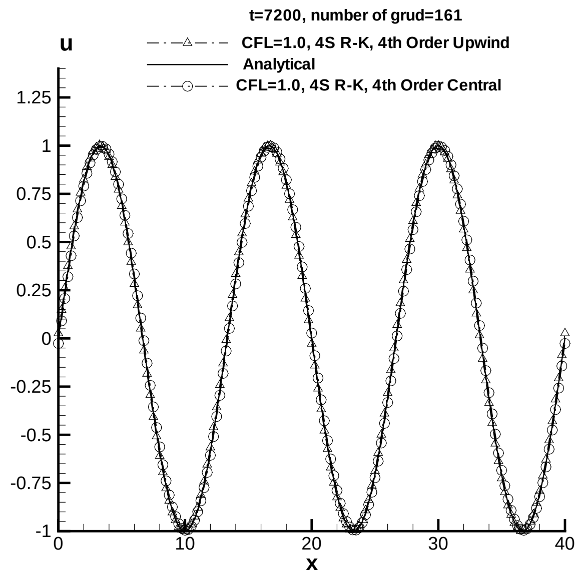

Figure 21: Numerical solutions of the wave equation for 4-Stage Runge-Kutta method with 4th order central and biased upwind differencing at refined grid, t=7200

Scheme	Stability	Dissipation Free	Dispersion Independent
2nd order Lax-Wendroff	CFL=1.0	CFL=1.0	none
2-stage R-K 2nd order central	none	none	none
2-stage R-K AOSD	CFL=1.0	CFL=1.0	none
4-stage R-K 2nd order central	CFL≤2.83	CFL≤1.0	CFL≤2.0
4-stage R-K AOSD	CFL≤1.73	none	none
4-stage R-K 2nd order upwind	CFL≤0.7	none	all
4-stage R-K 3rd order upwind	CFL≤1.75	none	all
4-stage R-K 4th order central	CFL≤2.06	CFL≤0.8	CFL≤1.5
4-stage R-K 4th order upwind	CFL≤1.05	none	all

Table 1: Summary of different schemes for the CFL limit of stability, dissipation free, and dispersion independent range