AIAA Paper 2004-2707

A Low Diffusion E-CUSP Upwind Scheme for Transonic Flows

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Objective:

• Develop an E-CUSP upwind scheme with high accuracy and efficiency

Background:

• Aircraft and engine design need CFD solver with high efficiency and accuracy

• Roe scheme popular for transonic flows with high resolution for discontinuities, matrix dissipation CPU intensive

• More efficient schemes with scalar dissipation:

H-CUSP schemes: Liou's AUSM family scheme, Edwards' LDFSS schemes, Van Leer-Hänel scheme, Jameson's H-CUSP schemes

E-CUSP: Jameson's H-CUSP schemes, Zha-Hu scheme(2004)

Flux Vector schemes: Steger-Warming scheme, Van Leer scheme; very diffusive

- H-CUSP schemes (e.g. AUSM family schemes) have high accuracy, but not fully consistent with characteristics

- E-CUSP scheme is consistent with characteristics. Zha-Hu E-CUSP scheme has high efficiency and low diffusion, able to capture exact contact surface. Non-smooth temperature field may occur.

- This paper is to remedy the Zha-Hu E-CUSP scheme to remove temperature oscillation.

Governing Equations

Quasi-1D Euler equations

$$\partial_t \mathbf{U} + \partial_x \mathbf{E} - \mathbf{H} = 0 \tag{1}$$

where $\mathbf{U} = S\mathbf{Q}$, $\mathbf{Q} = \begin{pmatrix} \rho \\ \rho u \\ \rho e \end{pmatrix}$, $\mathbf{E} = S\mathbf{F}$,

$$\mathbf{F} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (\rho e + p)u \end{pmatrix}, \quad \mathbf{H} = \frac{dS}{dx} \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} \tag{2}$$

Explicit finite volume method

$$\Delta \mathbf{Q}_i^{n+1} = \Delta t[-C(\mathbf{E}_{i+\frac{1}{2}} - \mathbf{E}_{i-\frac{1}{2}}) + \frac{\mathbf{H}_i}{S_i}]^n \tag{3}$$

Characteristics

Jacobian matrix

$$\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{Q}} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-\mathbf{1}} \tag{4}$$

where $\mathbf{T} = \begin{pmatrix} 1 & 1 & 1 \\ u-a & u & u+a \\ H-ua & \frac{1}{2}u^2 & H+ua \end{pmatrix}$

and

$$\mathbf{\Lambda} = \begin{pmatrix} u-a & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u+a \end{pmatrix} \tag{5}$$

Flux Splitting

$$\mathbf{F} = \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}\mathbf{Q} \tag{6}$$

$$\begin{aligned}\mathbf{F} &= \mathbf{T}\begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix}\mathbf{T}^{-1}\mathbf{Q} + \mathbf{T}\begin{pmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}\mathbf{T}^{-1}\mathbf{Q} \\ &= \mathbf{F^c} + \mathbf{F^p} \end{aligned} \tag{7}$$

where

$$\mathbf{F^c} = u\begin{pmatrix} \rho \\ \rho u \\ \rho e \end{pmatrix}, \mathbf{F^p} = \begin{pmatrix} 0 \\ p \\ pu \end{pmatrix} \tag{8}$$

$\mathbf{F^c}$ has eigenvalues (u, u, u), convective term, upwind

$\mathbf{F^p}$ has eigenvalues $(-a, 0, a)$, acoustic wave (pressure) term, upwind and downwind.

This splitting naturally leads to E-CUSP.

H-CUSP

$$\mathbf{F} = \mathbf{F}'^{\mathbf{c}} + \mathbf{F}'^{\mathbf{p}} \tag{9}$$

$$\mathbf{F}'^{\mathbf{c}} = u \begin{pmatrix} \rho \\ \rho u \\ \rho H \end{pmatrix}, \quad \mathbf{F}'^{\mathbf{p}} = \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} \tag{10}$$

where H is the total enthalpy

$$H = \frac{\rho e + p}{\rho} \tag{11}$$

$\mathbf{F}'^{\mathbf{c}}$ has eigenvalues $(u, u, \gamma u)$, upwind

$\mathbf{F}'^{\mathbf{p}}$ has eigenvalues $(0, 0, -(\gamma - 1)u)$, downwind

Zha-Hu E-CUSP Scheme

For $|u| \leq a$,

$$\mathbf{F}_{\frac{1}{2}} = \frac{1}{2}[(\rho u)_{\frac{1}{2}}(\mathbf{q^c}_L + \mathbf{q^c}_R) - |\rho u|_{\frac{1}{2}}(\mathbf{q^c}_R - \mathbf{q^c}_L)] + \begin{pmatrix} 0 \\ \mathcal{P}^+ p \\ \frac{1}{2}p(u + a_{\frac{1}{2}}) \end{pmatrix}_L + \begin{pmatrix} 0 \\ \mathcal{P}^- p \\ \frac{1}{2}p(u - a_{\frac{1}{2}}) \end{pmatrix}_R \quad (12)$$

For $u > a$, $\mathbf{F}_{\frac{1}{2}} = \mathbf{F}_L$; For $u < -a$, $\mathbf{F}_{\frac{1}{2}} = \mathbf{F}_R$

Interface mass flux is introduced based on Wada-Liou AUSMD scheme:

$$(\rho u)_{\frac{1}{2}} = (\rho_L u_L^+ + \rho_R u_R^-) \quad (13)$$

$$u_L^+ = a_{\frac{1}{2}}\{\frac{M_L + |M_L|}{2} + \alpha_L[\frac{1}{4}(M_L + 1)^2 - \frac{M_L + |M_L|}{2}]\} \quad (14)$$

$$u_R^- = a_{\frac{1}{2}}\{\frac{M_R - |M_R|}{2} + \alpha_R[-\frac{1}{4}(M_R - 1)^2 - \frac{M_R - |M_R|}{2}]\} \quad (15)$$

Zha-Hu E-CUSP Scheme, continued

Interface speed of sound

$$a_{\frac{1}{2}} = \frac{1}{2}(a_L + a_R) \tag{16}$$

$$M_L = \frac{u_L}{a_{\frac{1}{2}}}, \quad M_R = \frac{u_R}{a_{\frac{1}{2}}} \tag{17}$$

$$\alpha_L = \frac{2(p/\rho)_L}{(p/\rho)_L + (p/\rho)_R}, \quad \alpha_R = \frac{2(p/\rho)_R}{(p/\rho)_L + (p/\rho)_R} \tag{18}$$

Pressure splitting in momentum eq.

$$\mathcal{P}^{\pm} = \frac{1}{4}(M \pm 1)^2(2 \mp M) \pm \alpha M(M^2 - 1)^2, \quad \alpha = \frac{3}{16} \tag{19}$$

The Modified Scheme to Remove Temperature Oscillation

For energy equation:

$$\alpha_L = \frac{2(h_t/\rho)_L}{(h_t/\rho)_L + (h_t/\rho)_R}, \quad \alpha_R = \frac{2(h_t/\rho)_R}{(h_t/\rho)_L + (h_t/\rho)_R} \tag{20}$$

The total enthalpy:

$$h_t = e + \frac{p}{\rho} \tag{21}$$

Everything else is the same as the original Zha-Hu scheme.

Numerical Dissipation

At stagnation $u = 0$, the dissipation of the new scheme:

$$\mathbf{D} = -\frac{a_{\frac{1}{2}}}{2} \begin{pmatrix} 0 \\ 0 \\ \delta p \end{pmatrix} \tag{22}$$

where

$$\delta p = p_R - p_L \tag{23}$$

The dissipation of the Roe scheme:

$$\mathbf{D}_{Roe} = -\frac{\tilde{a}_{\frac{1}{2}}}{2(\gamma - 1)} \begin{pmatrix} (\gamma - 1)/\tilde{a}^2_{\frac{1}{2}} \delta p \\ 0 \\ \delta p \end{pmatrix} \tag{24}$$

The dissipation of the new scheme is not greater than that of the Roe scheme.

The Sod Shock Tube Problem

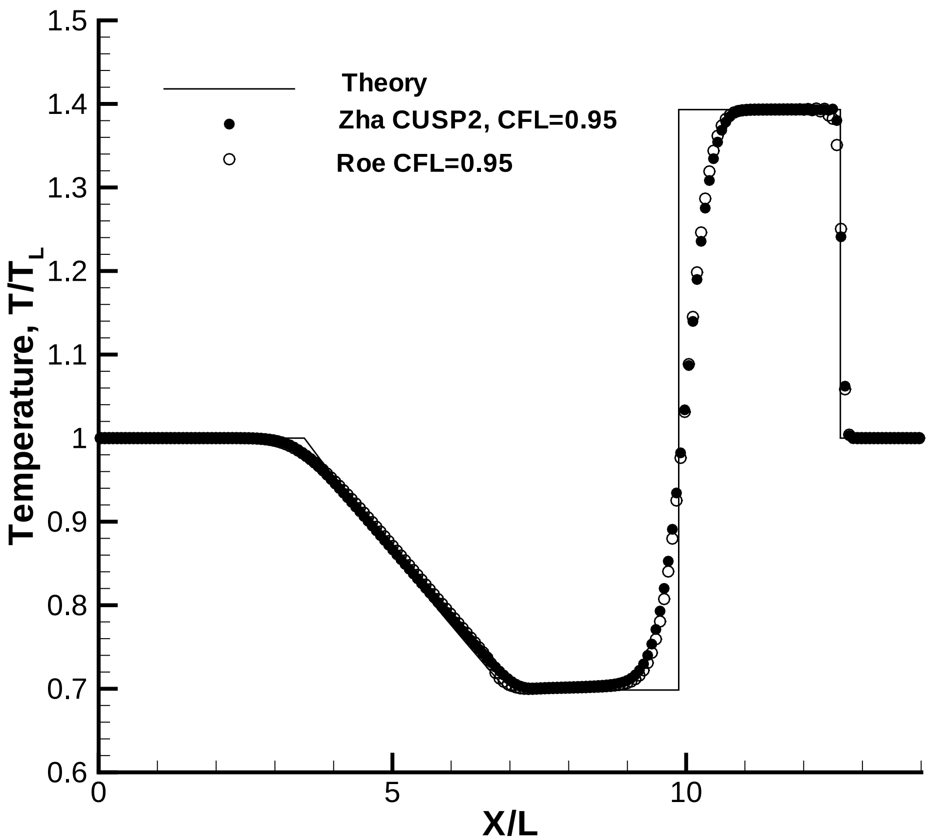


Figure 1: **Temperature**

The Sod Shock Tube Problem

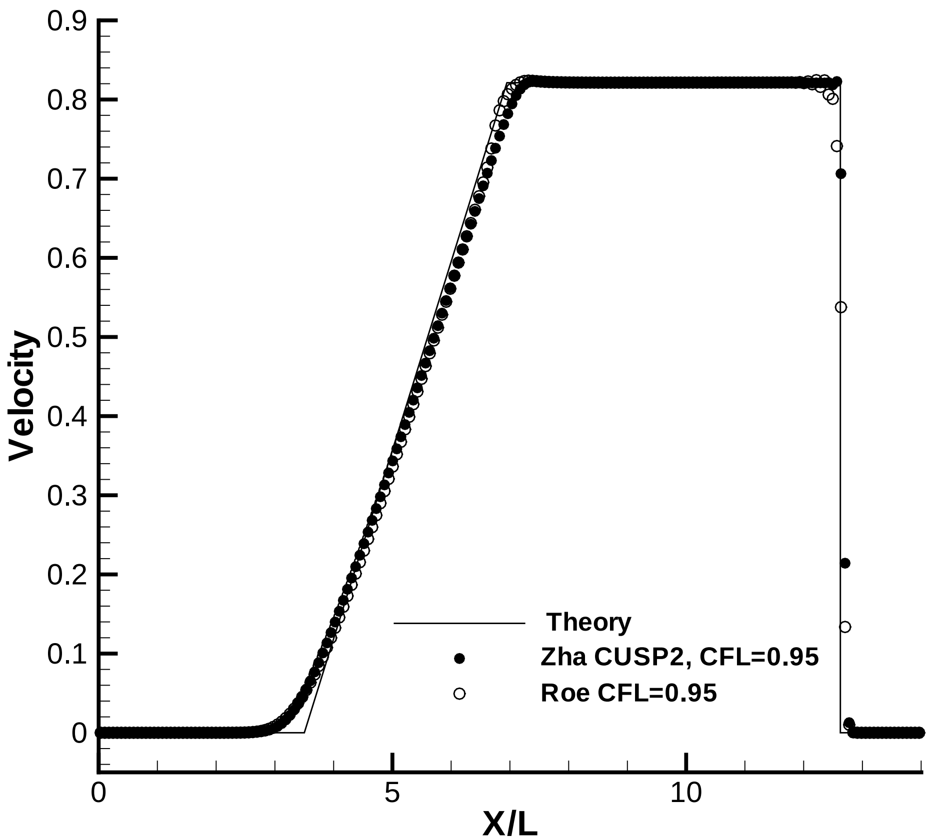


Figure 2: **Velocity**

The Sod Shock Tube Problem

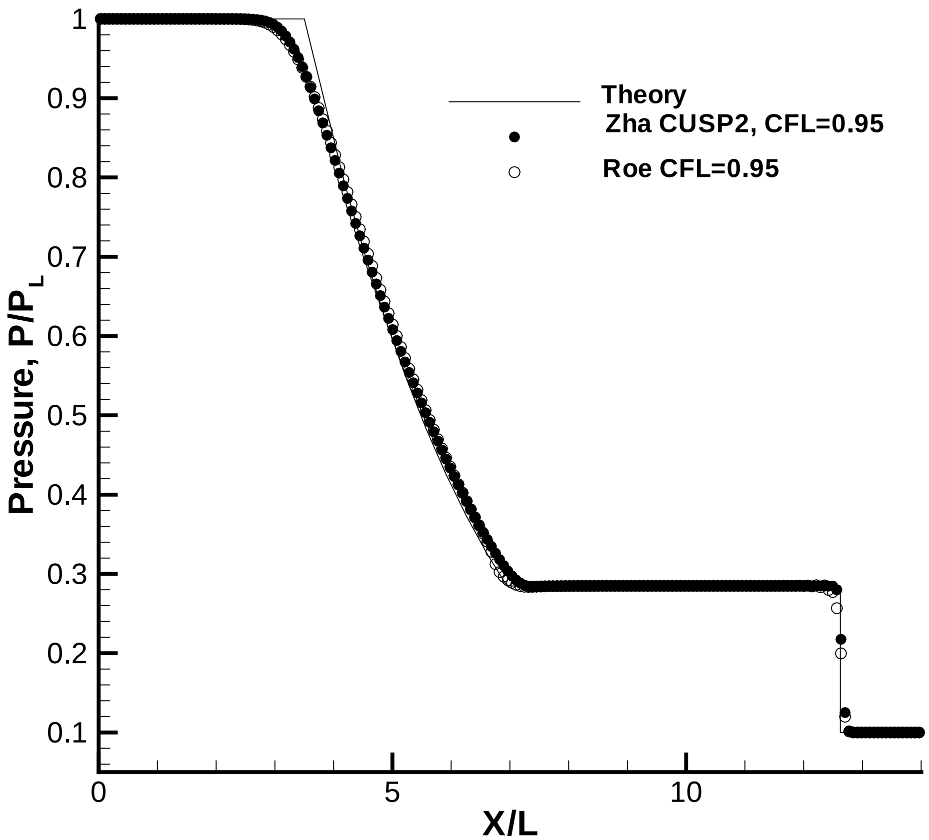
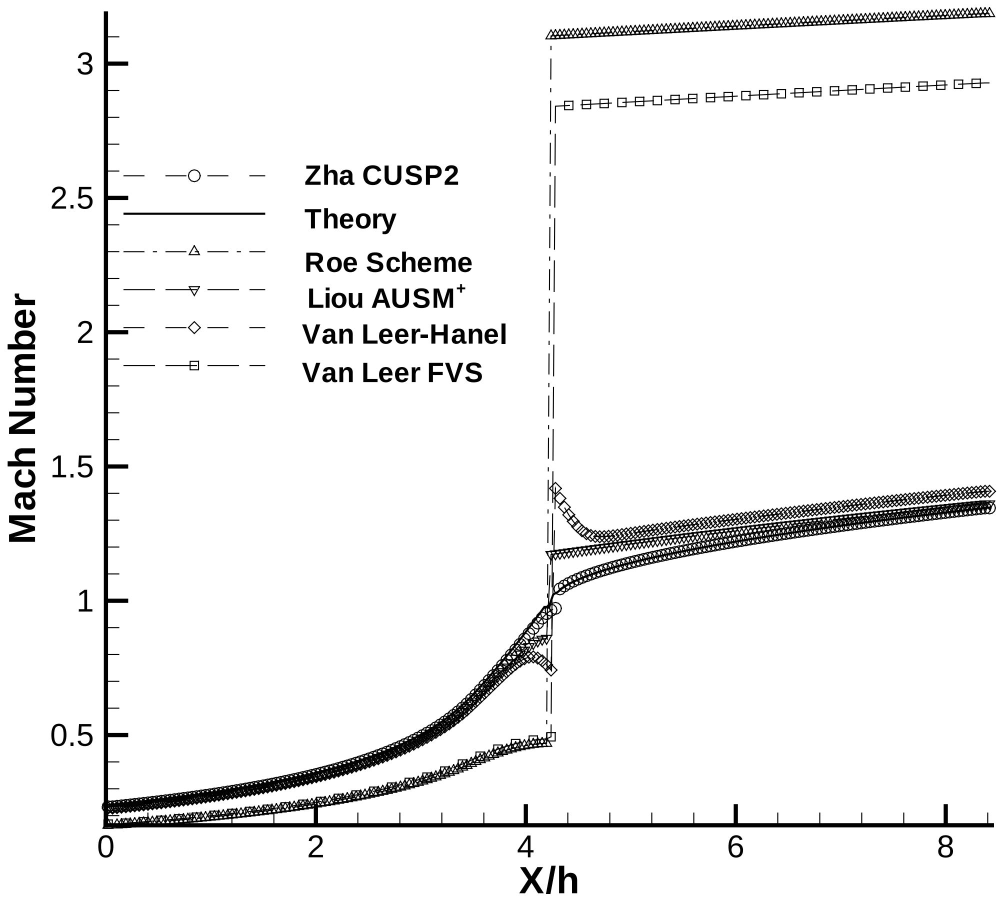
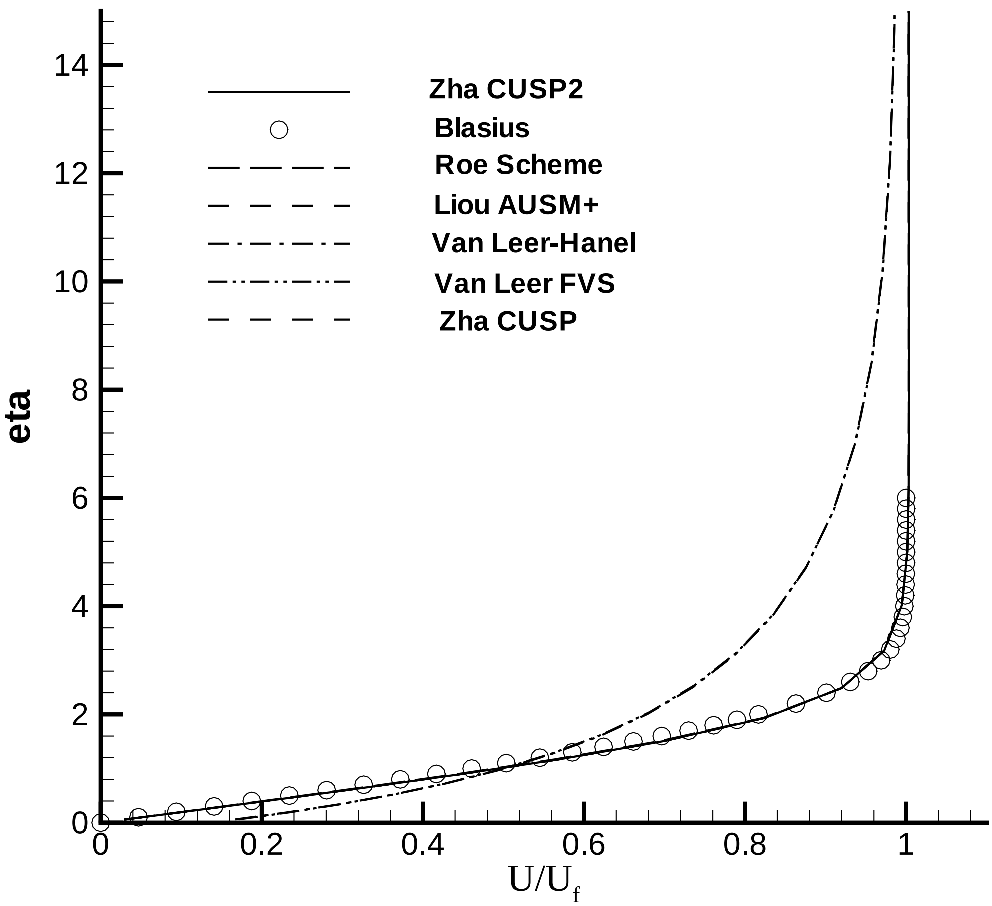


Figure 3: **Pressure**

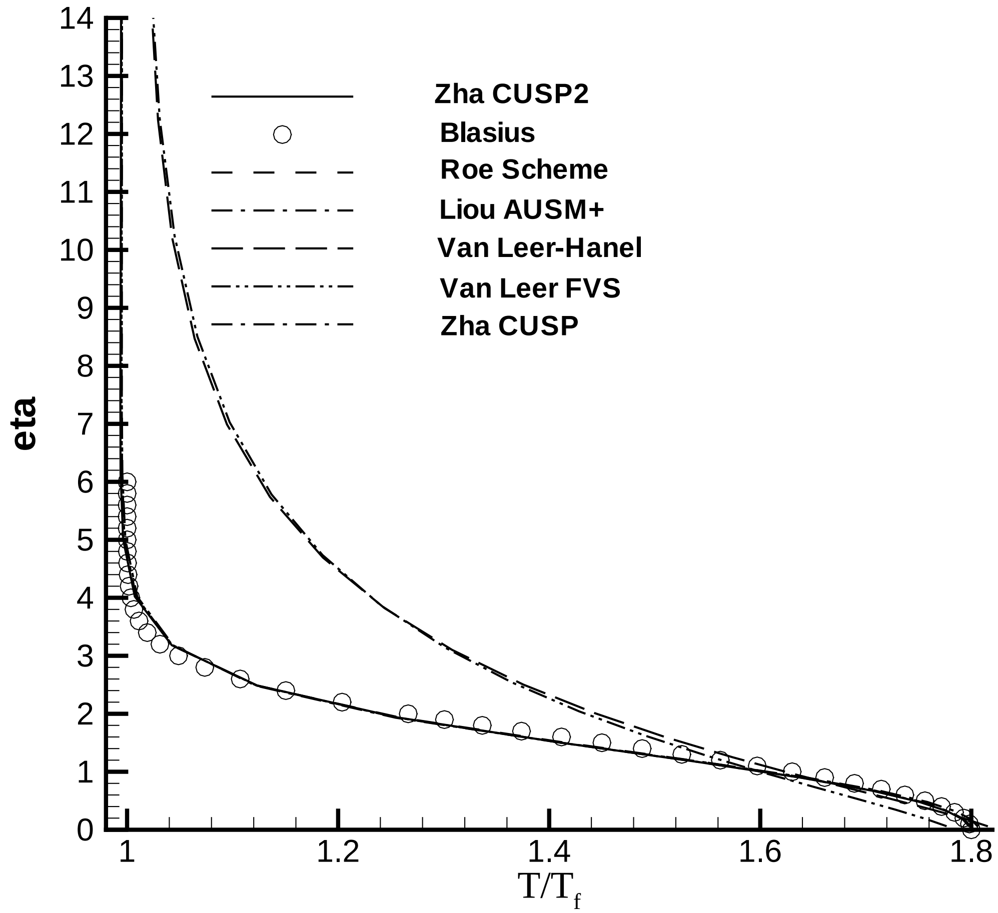
Quasi-1D Nozzle, Mach number



Laminar Flat Plate, M=2.0, Velocity Profile



Laminar Flat Plate, M=2.0, Temperature Profile

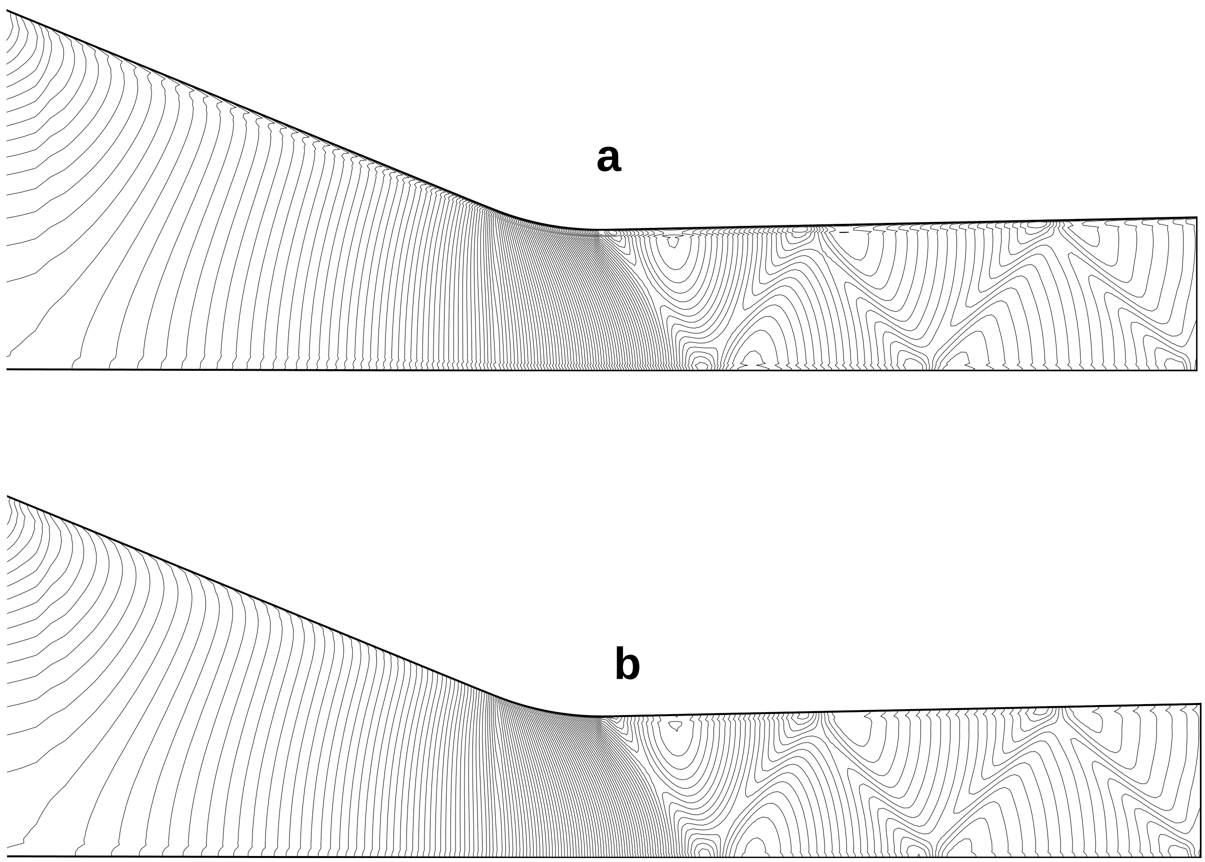


Laminar Flat Plate, Temperature Comparison of Different Schemes

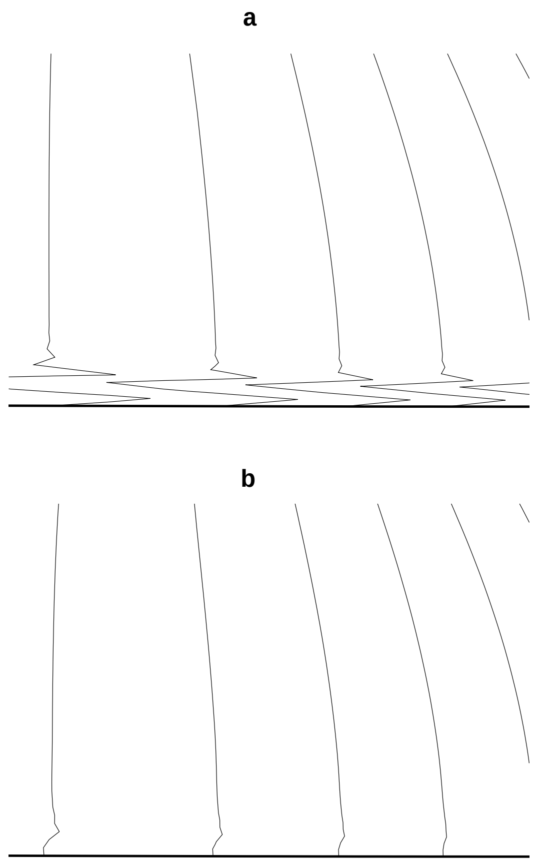
Scheme	40×30	80×60	160×80	error
Blasius	1.8000	1.8000	1.8000	0.0
Zha CUSP	1.8061	1.8022	1.8018	0.1%
Zha CUSP2	1.7980	1.7991	1.7988	-0.06%
Roe scheme	1.7990	1.8002	1.7996	-0.02%
Liou $AUSM^+$	1.7993	1.8000	1.8000	0.0
Van Leer	1.8157	1.8328	1.8333	1.8%
Van Leer-Hänel	1.7766	1.7970	1.7996	-0.02%

Table 1: Computed non-dimensional wall temperature using first order schemes with the baseline mesh and refined meshes

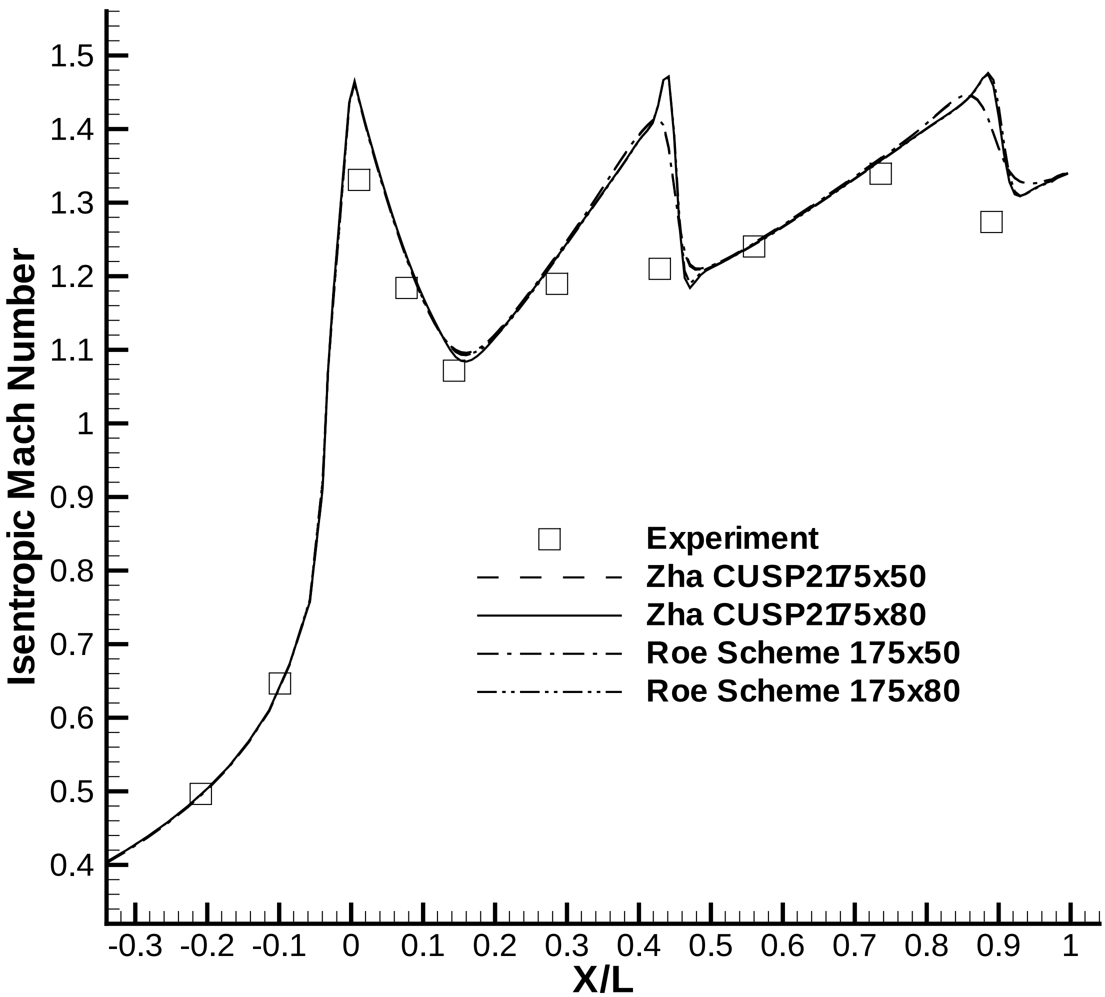
NASA Transonic Nozzle, Mach Number Contours



NASA Transonic Nozzle, Zoomed Near Wall Temperature Contours



Nozzle Isentropic Mach Number Distribution



Conclusions:

- The modified E-CUSP scheme removes the temperature oscillation

- The pressure term in the energy equation dissipation is replaced by the total enthalpy.

- The modified E-CUSP scheme is efficient and has low diffusion

- For 1D Sod shock problem, crisp shock profile achieved

- For quasi-1D nozzle, no expansion shock generated at sonic point.

- For M=2 laminar flat plate, 1st order scheme obtains accurate velocity and temperature profiles

- For a transonic nozzle, oblique shock captured well, temperature oscillation removed