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On the Accuracy of Runge-Kutta Methods for Unsteady Linear Wave Equation

Ge-Cheng Zha and Chakradhar Lingamgunta

Dept. of Mechanical Engineering
University of Miami
Coral Gables, Florida 33124
E-mail: zha@apollo.eng.miami.edu

Objective:

• Study the accuracy of 4-Stage Runge-Kutta Method for Unsteady CFD Calculation

Background

• Explicit 4-Stage Runge-Kutta method widely used for LES, DNS, CAA

• Unsteady Accuracy not well understood: Stability, Dissipation, Dispersion

Linear Wave Equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{1}$$

$$\frac{du}{dt} = R(u) \tag{2}$$

where

$$R(u) = -c\frac{\partial u}{\partial x} \tag{3}$$

$$c = 1$$

Initial solution:

$$u(x,0) = sin2n\pi(\frac{x}{40}), \qquad 0 \le x \le 40. \tag{4}$$

The analytical solution with periodic boundary conditions:

$$u(x,t) = sin2n\pi(\frac{x-t}{40}), \qquad 0 \le x \le 40. \tag{5}$$

Runge-Kutta Methods (R-K):

Multistage R-K matches Taylor-series expansion

$$u^{n+1} = u^n + \Delta t u_t^n + \frac{(\Delta t)^2}{2} u_{tt}^n + \frac{(\Delta t)^3}{6} u_{ttt}^n + \frac{(\Delta t)^4}{24} u_{tttt}^n + ... \quad (6)$$

Lax-Wendroff Scheme:

One Stage method:

$$u^{n+1} = u^n - c\Delta t u_x^n + \frac{(c\Delta t)^2}{2} u_{xx}^n - \frac{(c\Delta t)^3}{6} u_{xxx}^n + \frac{(c\Delta t)^4}{24} u_{xxxx}^n + ... \quad (7)$$

- For 1D linear wave eq., these two methods are equivalent.
- For multi-D nonlinear Euler or N.S eqs., high order Lax-Wendroff scheme very complicated.

2-Stage R-K:

Stage 1:

$$u^{(1)} = u^n + \Delta t R^{(n)} \tag{8}$$

Stage 2:

$$u^{n+1} = u^n + \frac{\Delta t}{2}(R^{(n)} + R^{(1)}) \tag{9}$$

4-Stage R-K:

Stage 1:

$$u^{(1)} = u^n + \frac{\Delta t}{2} R^{(n)} \tag{10}$$

Stage 2:

$$u^{(2)} = u^n + \frac{\Delta t}{2} R^{(1)} \tag{11}$$

Stage 3:

$$u^{(3)} = u^n + \Delta t R^{(2)} \tag{12}$$

Stage 4:

$$u^{n+1} = u^n + \frac{\Delta t}{6}(R^{(n)} + 2R^{(1)} + 2R^{(2)} + R^{(3)}) \tag{13}$$

Schemes Studied:

1) 2nd order Lax-Wendroff Scheme (used as reference)

$$u_i^{n+1} = u_i^n - \frac{\nu}{2}(u_{i+1}^n - u_{i-1}^n) + \frac{\nu^2}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \quad (14)$$

where ν is the CFL number expressed as:

$$\nu = \frac{c\Delta t}{\Delta x} \quad (15)$$

Stability $CFL \leq 1$,

When $CFL = 1$, no dissipation and dispersion,

When $CFL < 1$, very large dissipation and dispersion

2) 2-stage R-K, 2nd order central differencing

$$R_i^{(n)} = -c\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \tag{16}$$

This scheme is unstable.

3) 2-stage R-K, 1st order Alternating One-Side Differencing (AOSD)

McCormack Scheme

Stage 1, Predictor:

$$R_i^{(n)} = -c\frac{u_{i+1}^n - u_i^n}{\Delta x} \quad (downwind) \tag{17}$$

Stage 2, Corrector:

$$R_i^{(1)} = -c\frac{u_i^{(1)} - u_{i-1}^{(1)}}{\Delta x} \quad (upwind) \tag{18}$$

This scheme is the same as Lax-Wendroff Scheme

4) 4-stage R-K, 2nd order central differencing

Stability: $CFL \leq 2.83$

Dissipation free: $CFL \leq 1.0$

Dispersion invariant: $CFL \leq 2.0$

5) 4-stage R-K, 1st Order AOSD (2nd order spatial accuracy)

Stability: $CFL \leq 1.73$

Dissipation free: none

Dispersion invariant: none

6) 4-stage R-K, 2nd order upwind differencing

$$R_i^{(n)} = -c(\frac{3u_i^n - 4u_{i-1}^n + u_{i-2}^n}{2\Delta x}) \tag{19}$$

Stability: $CFL \leq 0.7$

Dissipation free: none

Dispersion and dissipation invariant: $CFL \leq 0.7$

7) 4-stage R-K, 3rd order biased upwind differencing

$$R_i^{(n)} = -c(\frac{2u_{i+1}^n + 3u_i^n - 6u_{i-1}^n + u_{i-2}^n}{6\Delta x}) \tag{20}$$

Stability: $CFL \leq 1.75$

Dissipation free: none

Dispersion and dissipation invariant: $CFL \leq 1.75$

8) 4-stage R-K, 4th order biased upwind differencing

$$R_i^{(n)} = -c(\frac{3u_{i+1}^n + 10u_i^n - 18u_{i-1}^n + 6u_{i-2}^n - u_{i-3}^n}{12\Delta x}) \tag{21}$$

Stability: $CFL \leq 1.05$

Dissipation free: none

Dispersion and dissipation invariant: $CFL \leq 1.05$

9) 4-stage R-K, 4th order central differencing

$$R_i^{(n)} = -c(\frac{-u_{i+2}^n + 8u_{i+1}^n - 8u_{i-1}^n + u_{i-2}^n}{12\Delta x}) \tag{22}$$

Stability: $CFL \leq 2.06$

Dissipation free: $CFL \leq 0.8$

Dispersion invariant: $CFL \leq 1.5$

For 1D linear wave eq., it is the same to represent the derivatives in the Lax-Wendroff scheme by:

$$u_x^n = \frac{-u_{i+2}^n + 8u_{i+1}^n - 8u_{i-1}^n + u_{i-2}^n}{12\Delta x} \tag{23}$$

$$u_{xx}^n = \frac{u_{i+4}^n - 16u_{i+3}^n + 64u_{i+2}^n + 16u_{i+1}^n}{144\Delta x^2} + \frac{-130u_i^n + 16u_{i-1}^n + 64u_{i-2}^n - 16u_{i-3}^n + u_{i-4}^n}{144\Delta x^2} \tag{24}$$

$$u^n_{xxx} = \frac{-u^n_{i+6} + 24u^n_{i+5} - 192u^n_{i+4} + 488u^n_{i+3}}{1728\Delta x^3}$$
$$+\frac{387u^n_{i+2} - 1584u^n_{i+1} + 1584u^n_{i-1} - 387u^n_{i-2}}{1728\Delta x^3}$$
$$+\frac{-488u^n_{i-3} + 192u^n_{i-4} - 24u^n_{i-5} + u^n_{i-6}}{1728\Delta x^3} \tag{25}$$

$$u^n_{xxxx} = \frac{u^n_{i+8} - 32u^n_{i+7} + 384u^n_{i+6} - 2016u^n_{i+5}}{20736\Delta x^4}$$
$$+\frac{3324u^n_{i+4} + 6240u^n_{i+3} - 16768u^n_{i+2} - 4192u^n_{i+1}}{20736\Delta x^4}$$
$$+\frac{26118u^n_{i} - 4192u^n_{i-1} - 16768u^n_{i-2} + 6240u^n_{i-3}}{20736\Delta x^4}$$
$$+\frac{3324u^n_{i-4} - 2016u^n_{i-5}}{20736\Delta x^4} + \frac{384u^n_{i-6} - 32u^n_{i-7} + u^n_{i-8}}{20736\Delta x^4} \tag{26}$$

2nd order Lax-Wendroff scheme

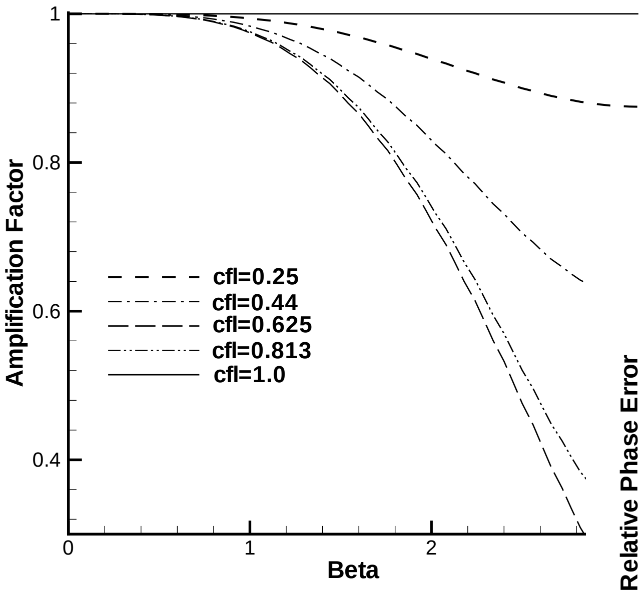

Figure 1: 2nd order Lax-Wendr scheme

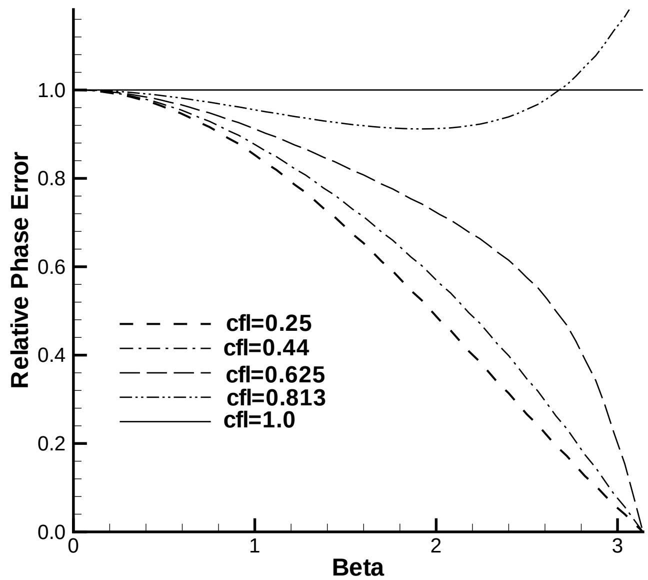

Figure 2: 2nd order Lax-Wendroff scheme

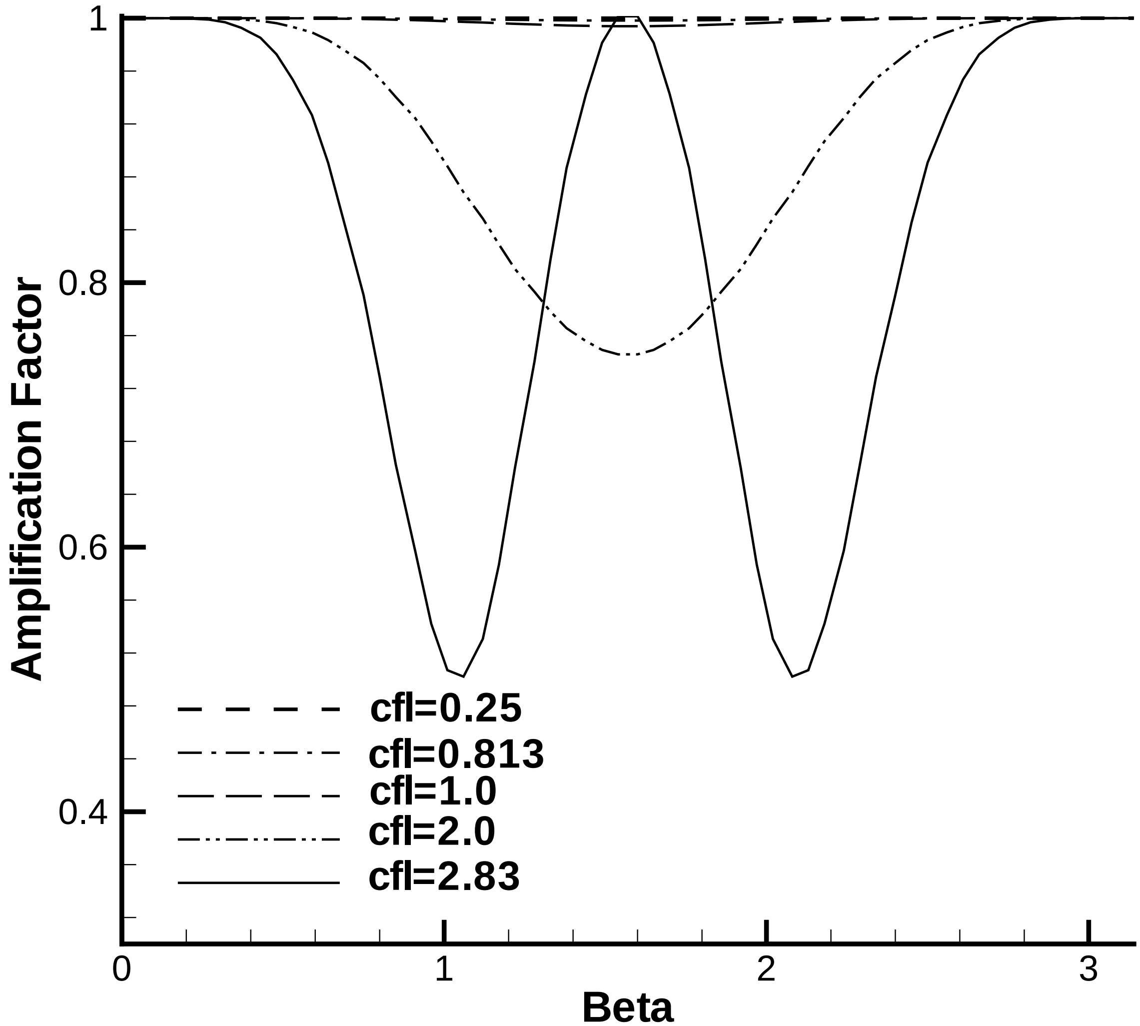

Figure 3: **4-Stage R-K, 2nd order central differencing**

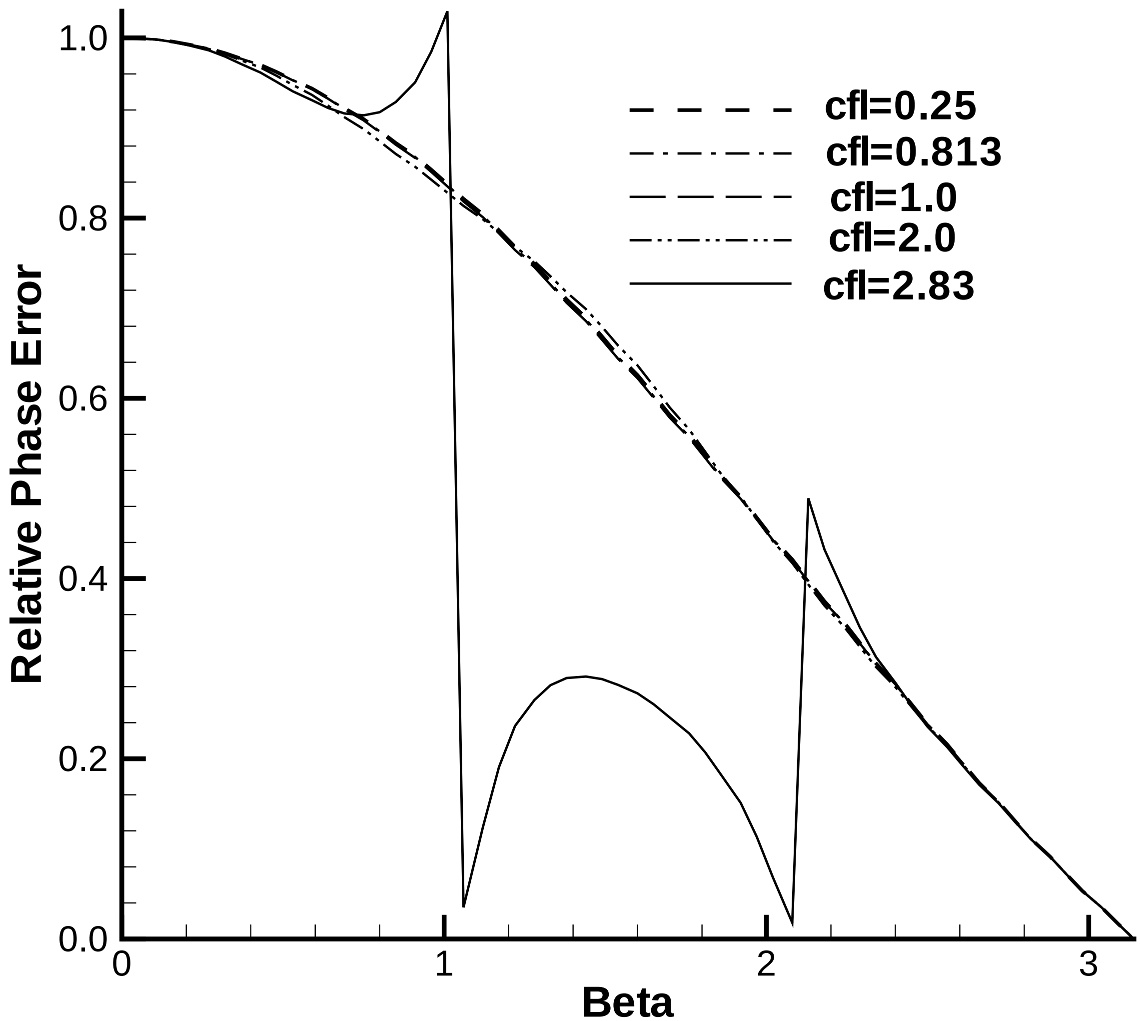

Figure 4: **4-Stage R-K, 2nd order central differencing**

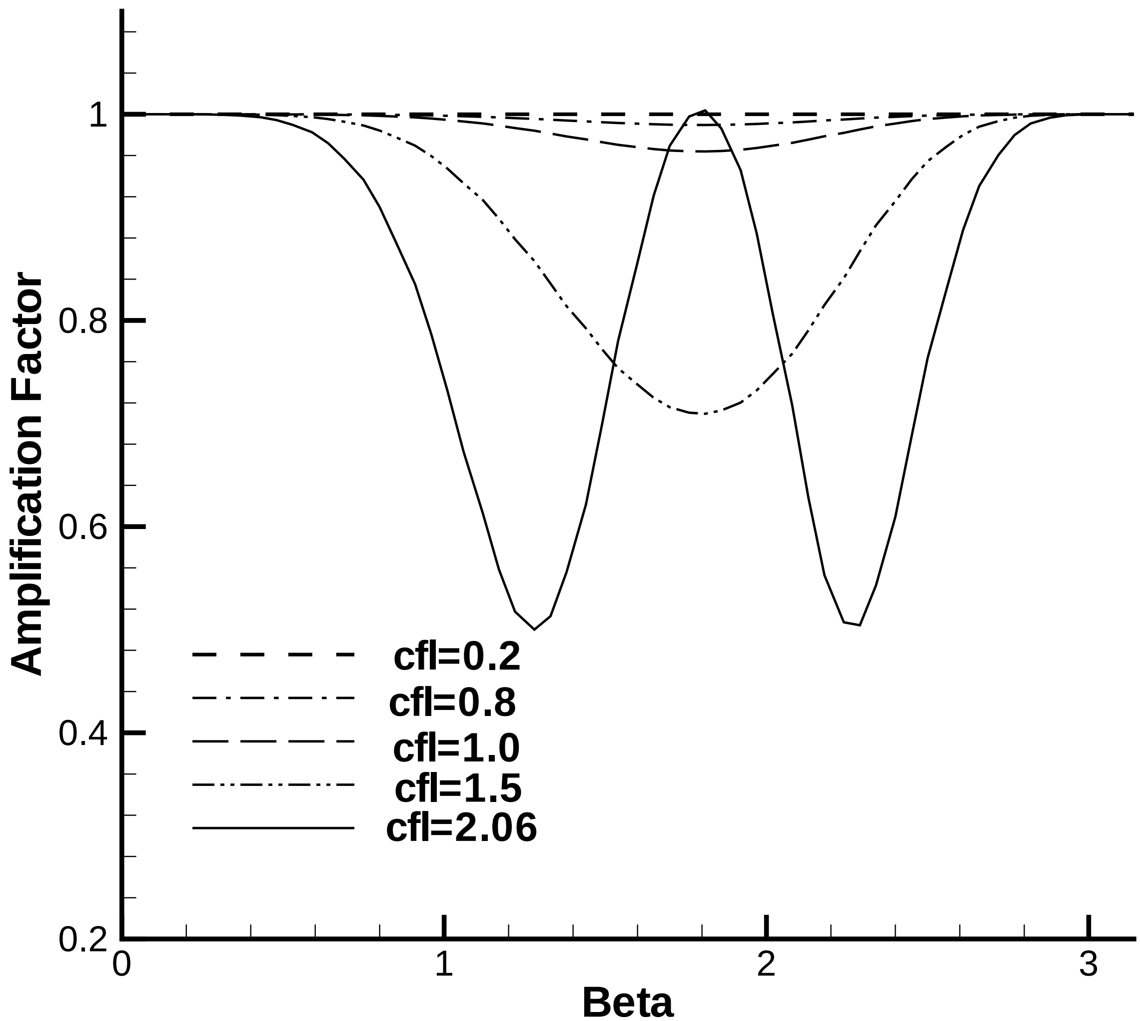

Figure 5: **4-Stage R-K, 4th order central differencing**

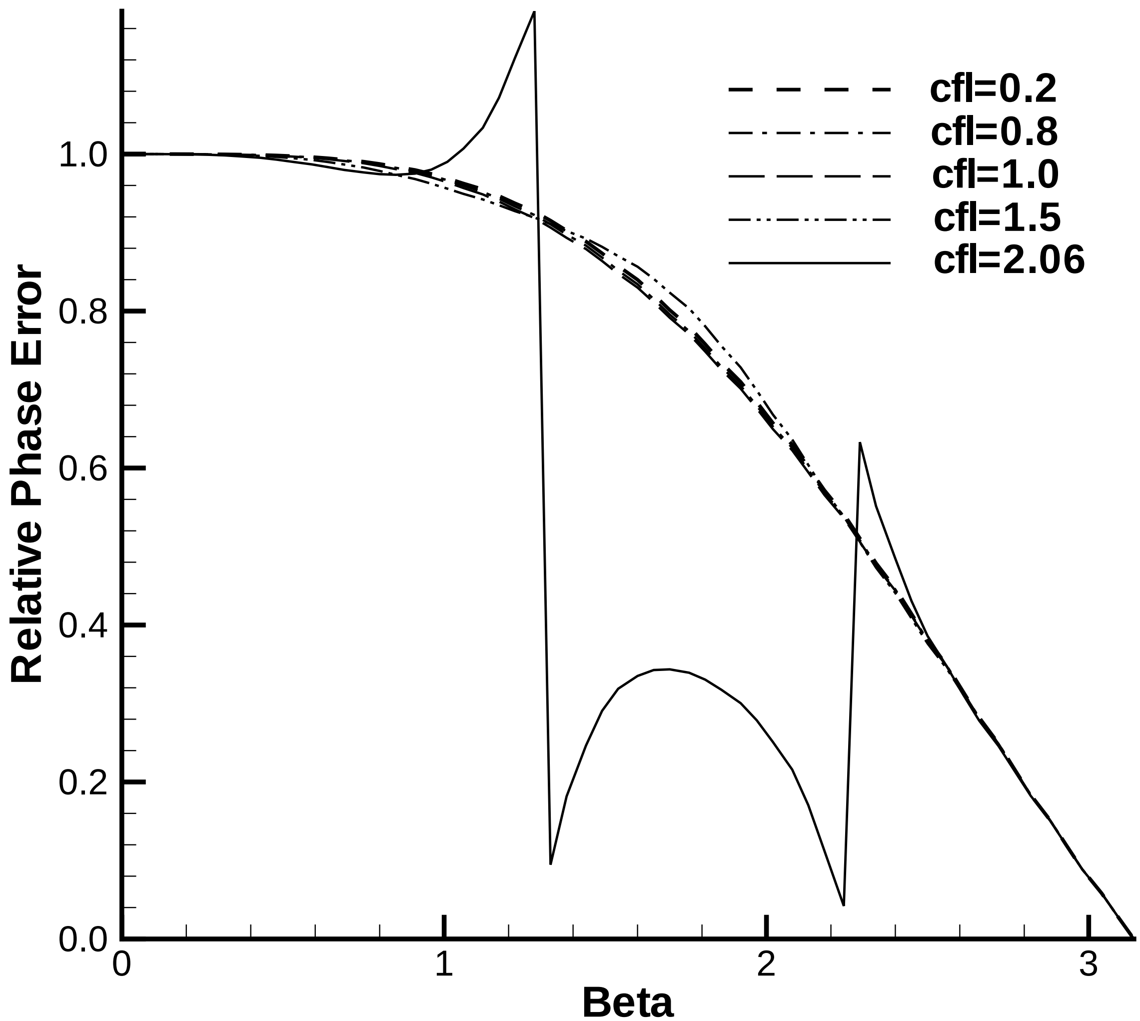

Figure 6: **4-Stage R-K, 4th order central differencing**

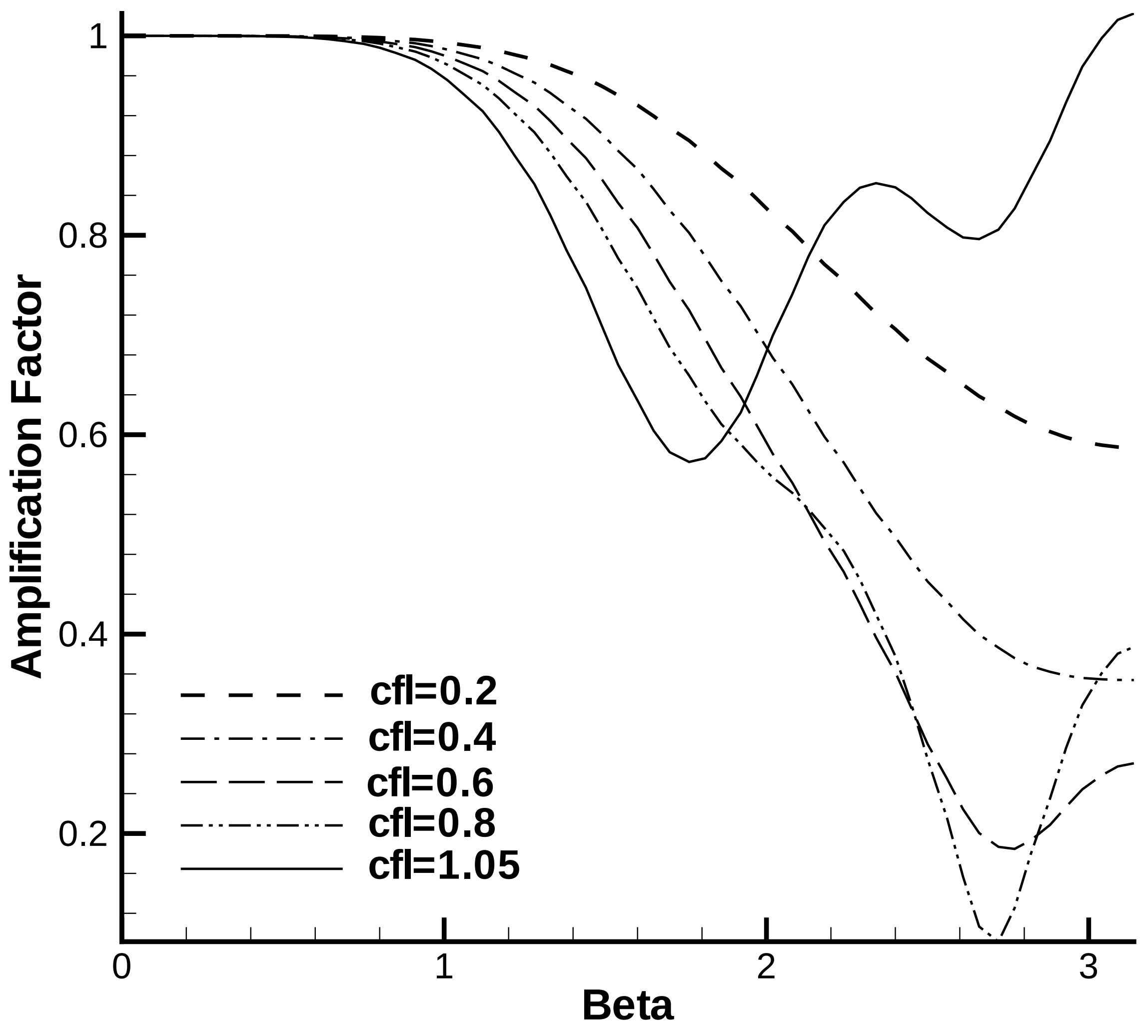

Figure 7: **4-Stage R-K, 4th order biased upwind differencing**

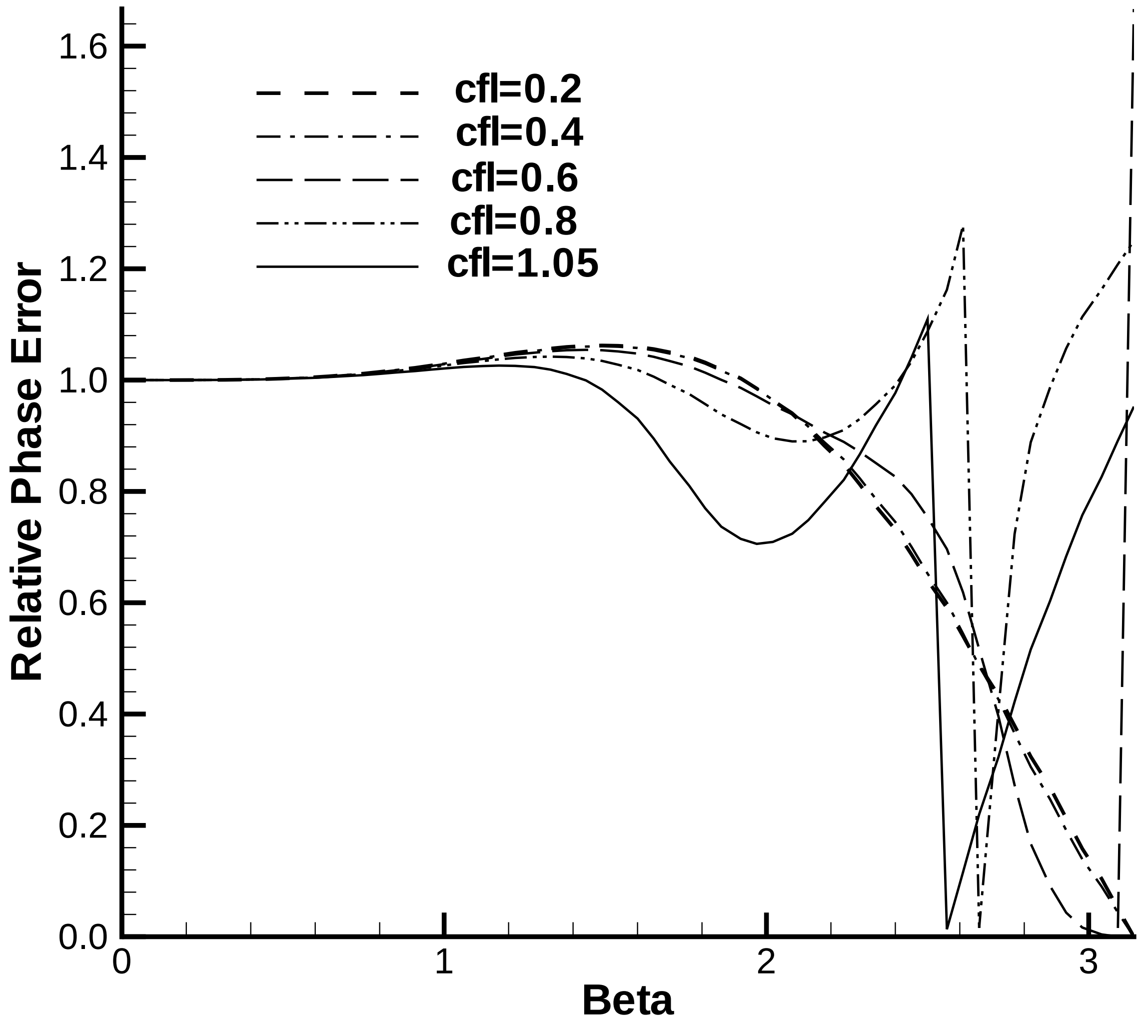

Figure 8: **4-Stage R-K, 4th order biased upwind differencing**

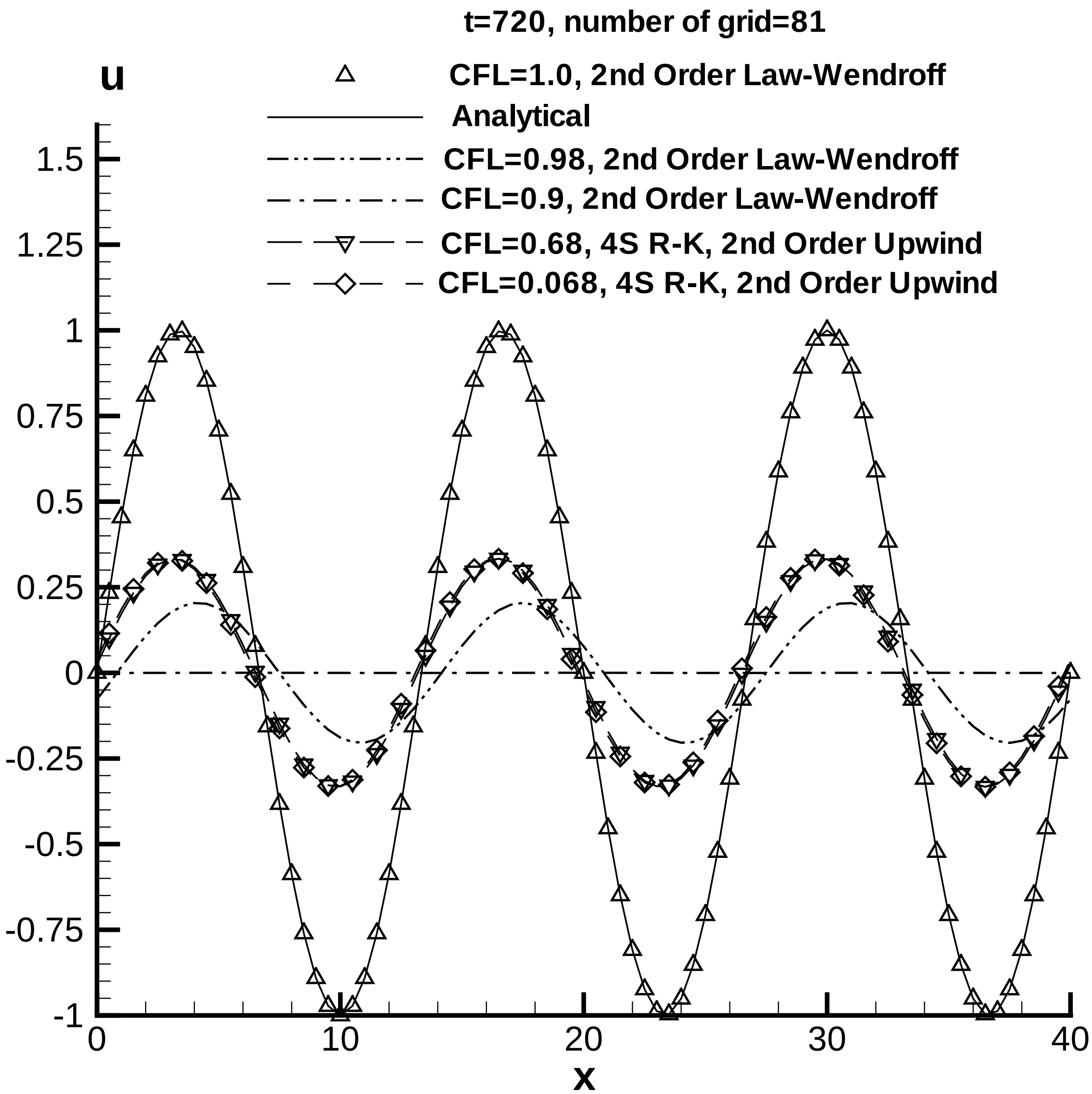

Figure 9: Numerical solutions of the wave equation for Lax-Wendroff scheme and 4-Stage Runge-Kutta method with 2nd order upwind differencing, t=720

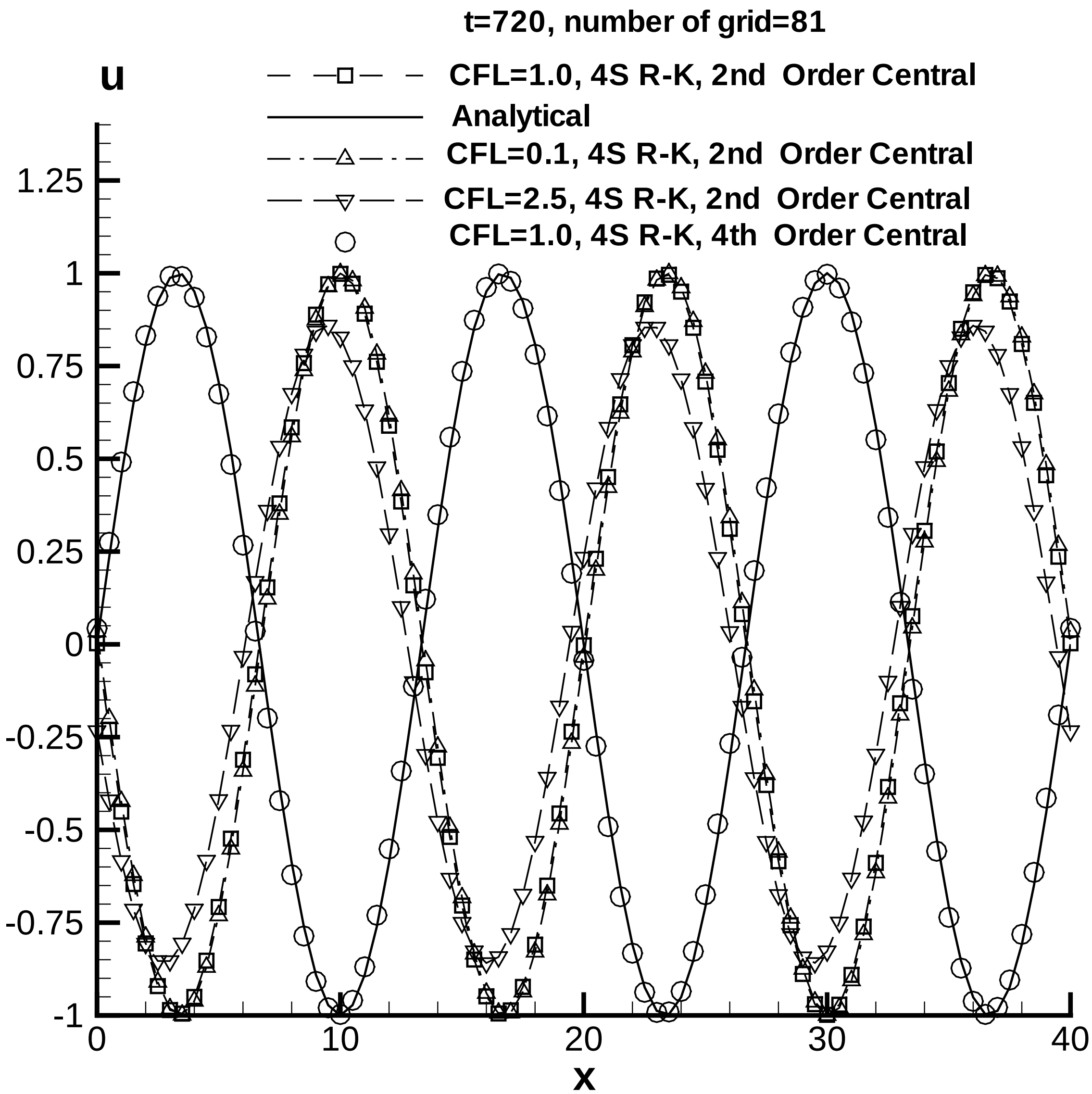

Figure 10: Numerical solutions of the wave equation for 4-Stage Runge-Kutta method with 2nd and 4th order central differencing, t=720

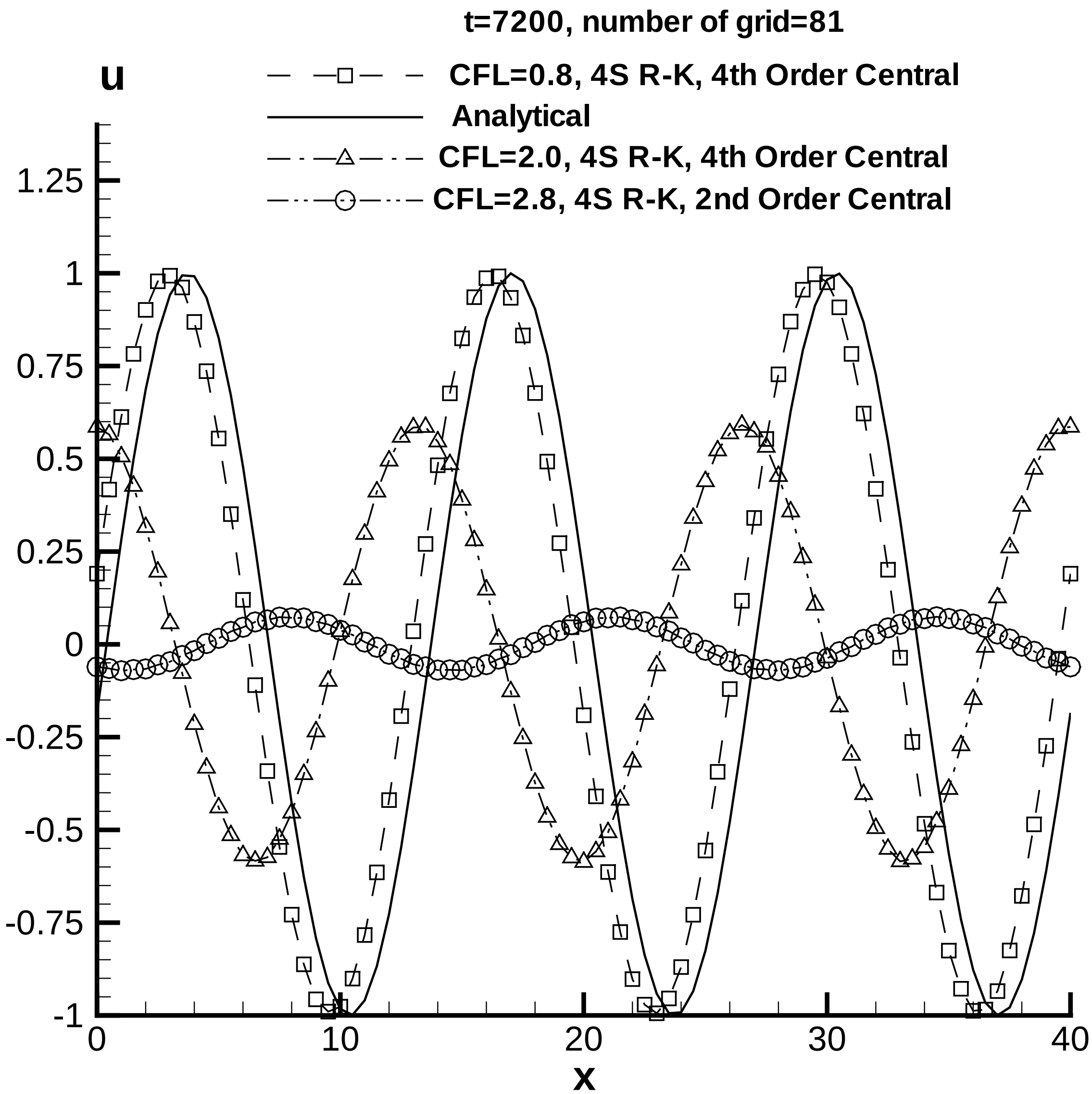

Figure 11: Numerical solutions of the wave equation for 4-Stage Runge-Kutta method with 2nd and 4th order order central differencing, t=7200.

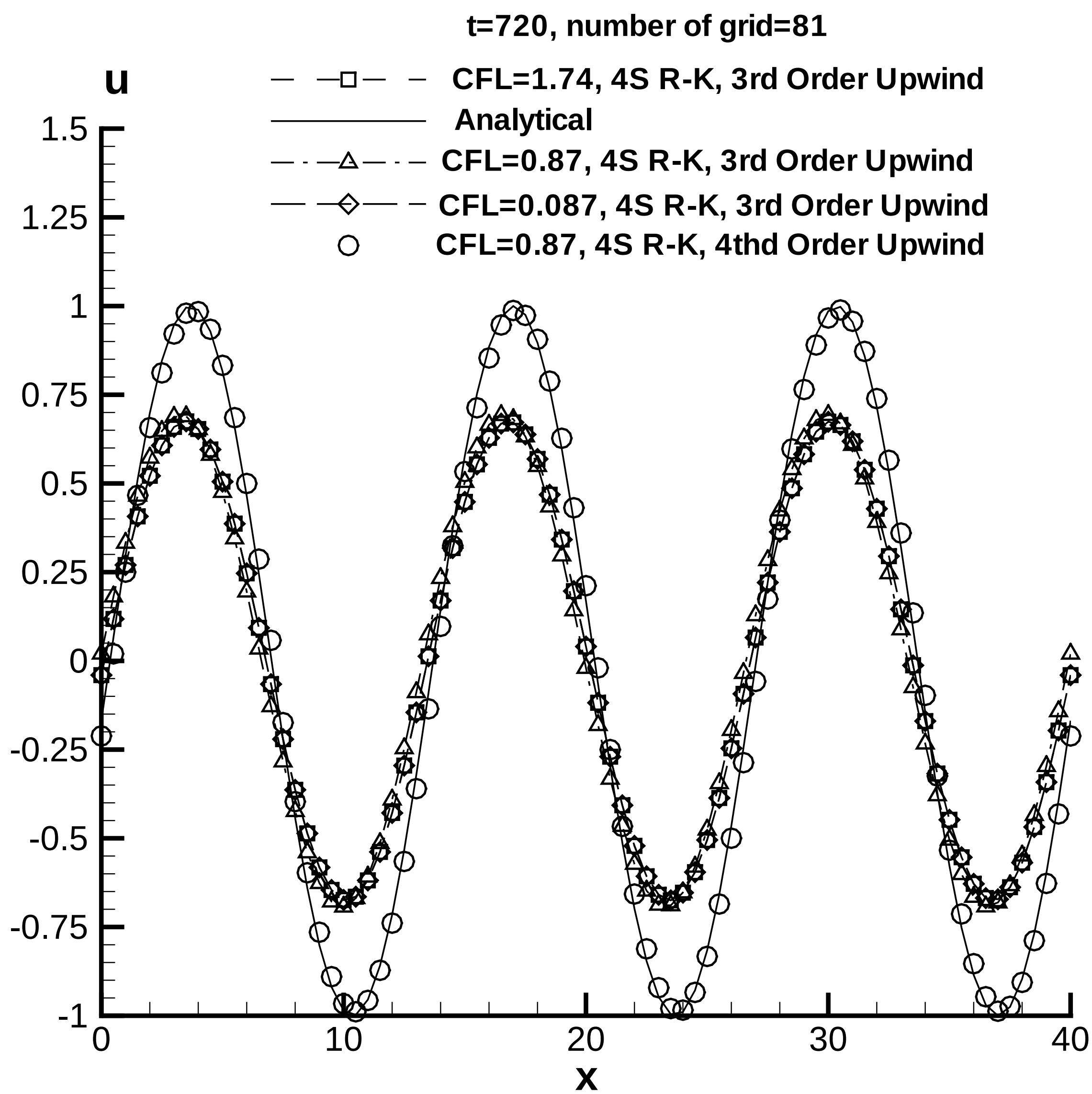

Figure 12: Numerical solutions of the wave equation for 4-Stage Runge-Kutta method with 3rd and 4th order biased Upwind differencing, t=720.

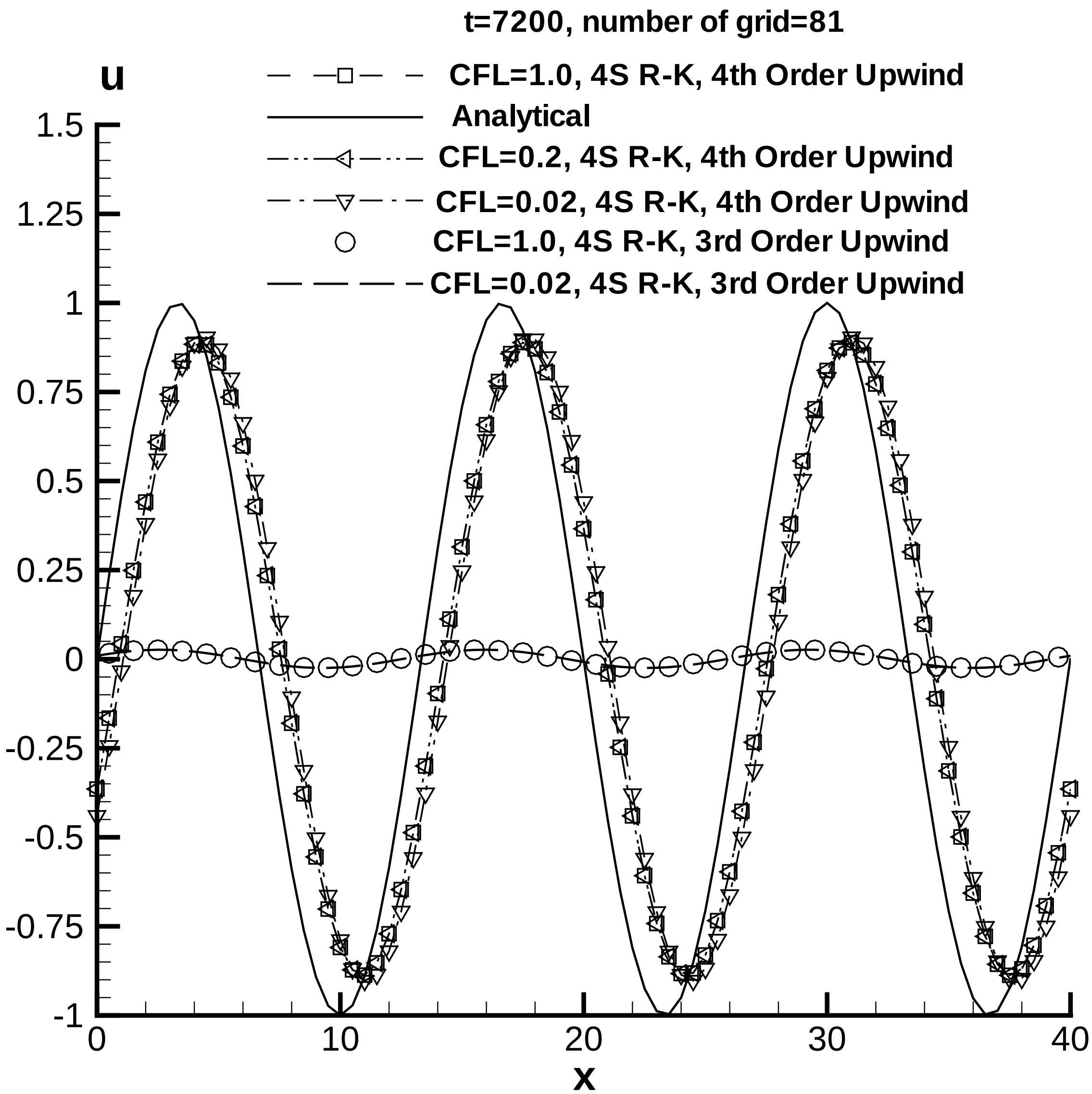

Figure 13: Numerical solutions of the wave equation for 4-Stage Runge-Kutta method with 3rd and 4th order biased Upwind differencing, t=720.

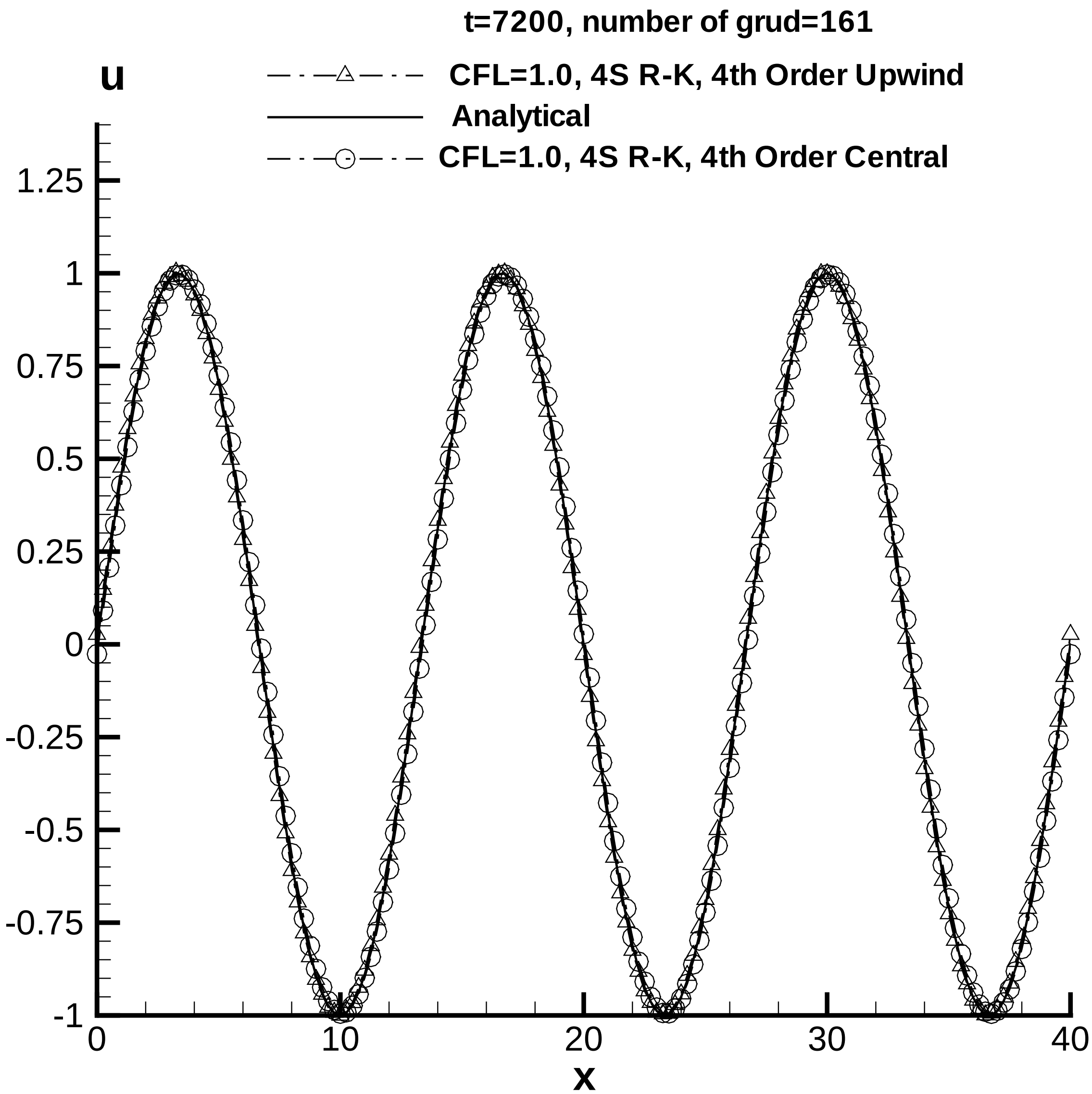

Figure 14: Numerical solutions of the wave equation for 4-Stage Runge-Kutta method with 4th order central and biased upwind differencing at refined grid, t=7200

CONCLUSIONS:

• 2nd order Lax-Wendroff scheme too diffusive
when $CFL < 1$ for unsteady calculation

• 2-stage R-K, 2nd order central differencing unstable

• 4-stage R-K, 2nd order central differencing:

stability: $CFL \leq 2.83$
dissipation free: $CFL \leq 1.0$
dispersion invariant: $CFL \leq 2.0$

• 4-stage R-K, 3rd order biased upwind differencing:

stability: $CFL \leq 1.75$
dissipation free: none
dispersion and dissipation invariant: $CFL \leq 1.75$

• 4-stage R-K, 4th order biased upwind differencing:

stability: $CFL \leq 1.05$
dissipation free: none
dispersion and dissipation invariant: $CFL \leq 1.05$

- 4-stage R-K, 2th order central differencing:

stability: $CFL \leq 2.06$
dissipation free: $CFL \leq 0.8$
dispersion invariant: $CFL \leq 1.5$

- Upwind schemes with 4-Stage R-K are less stable than central differencing

- For low frequency linear wave solutions, the 4th order biased upwind differencing and 4th order central differencing have the equivalent accuracy.