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Calculation of Transonic Internal Flows Using an Efficient High Resolution Upwind Scheme

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Objective:

- Develop an E-CUSP upwind scheme with high accuracy and efficiency

Background:

- Aircraft and engine design need CFD solver with high efficiency and accuracy
- Roe scheme popular for transonic flows with high resolution for discontinuities
- More efficient schemes with scalar dissipation:

H-CUSP schemes: Liou's AUSM family scheme, Edwards' LDFSS schemes, Van Leer-Hänel scheme, Jameson's H-CUSP schemes

E-CUSP: Jameson's H-CUSP schemes, Zha-Bilgen scheme(1993), Zha's scheme (1999)

Flux Vector schemes: Steger-Warming scheme, Van Leer scheme

- H-CUSP schemes (e.g. AUSM family schemes) have high accuracy, but not fully consistent with characteristics

- E-CUSP scheme is consistent with characteristics. Previous E-CUSP scheme is not smooth, or not able to capture the contact surfaces.

- This paper is to develop an E-CUSP scheme which is efficient, accurate and robust.

Governing Equations

Quasi-1D Euler equations

$$\partial_t \mathbf{U} + \partial_x \mathbf{E} - \mathbf{H} = 0 \tag{1}$$

where $\mathbf{U} = S\mathbf{Q}$, $\mathbf{Q} = \begin{pmatrix} \rho \\ \rho u \\ \rho e \end{pmatrix}$, $\mathbf{E} = S\mathbf{F}$,

$$\mathbf{F} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (\rho e + p)u \end{pmatrix}, \quad \mathbf{H} = \frac{dS}{dx} \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} \tag{2}$$

Explicit finite volume method

$$\Delta \mathbf{Q}_i^{n+1} = \Delta t[-C(\mathbf{E}_{i+\frac{1}{2}} - \mathbf{E}_{i-\frac{1}{2}}) + \frac{\mathbf{H}_i}{S_i}]^n \tag{3}$$

Characteristics

Jacobian matrix

$$\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{Q}} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-\mathbf{1}} \tag{4}$$

where $\mathbf{T} = \begin{pmatrix} 1 & 1 & 1 \\ u-a & u & u+a \\ H-ua & \frac{1}{2}u^2 & H+ua \end{pmatrix}$

and

$$\mathbf{\Lambda} = \begin{pmatrix} u-a & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u+a \end{pmatrix} \tag{5}$$

Flux Splitting

$$\mathbf{F} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}\mathbf{Q} \tag{6}$$

$$\begin{aligned}\mathbf{F} = \mathbf{T}\begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix}\mathbf{T}^{-1}\mathbf{Q} + \mathbf{T}\begin{pmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}\mathbf{T}^{-1}\mathbf{Q} \\ = \mathbf{F^c} + \mathbf{F^p}\end{aligned} \tag{7}$$

where

$$\mathbf{F^c} = u\begin{pmatrix} \rho \\ \rho u \\ \rho e \end{pmatrix}, \mathbf{F^p} = \begin{pmatrix} 0 \\ p \\ pu \end{pmatrix} \tag{8}$$

$\mathbf{F^c}$ has eigenvalues (u, u, u), convective term, upwind

$\mathbf{F^p}$ has eigenvalues $(-a, 0, a)$, acoustic wave (pressure) term, upwind and downwind.

This splitting naturally leads to E-CUSP.

H-CUSP

$$\mathbf{F} = \mathbf{F'^c} + \mathbf{F'^p} \tag{9}$$

$$\mathbf{F'^c} = u \begin{pmatrix} \rho \\ \rho u \\ \rho H \end{pmatrix}, \quad \mathbf{F'^p} = \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} \tag{10}$$

where H is the total enthalpy

$$H = \frac{\rho e + p}{\rho} \tag{11}$$

$\mathbf{F'^c}$ has eigenvalues $(u, u, \gamma u)$, upwind

$\mathbf{F'^p}$ has eigenvalues $(0, 0, -(\gamma - 1)u)$, downwind

The New E-CUSP Scheme

For $|u| \le a$,

$$\mathbf{F}_{\frac{1}{2}} = \frac{1}{2}[(\rho u)_{\frac{1}{2}}(\mathbf{q^c}_L + \mathbf{q^c}_R) - |\rho u|_{\frac{1}{2}}(\mathbf{q^c}_R - \mathbf{q^c}_L)] + \begin{pmatrix} 0 \\ \mathcal{P}^+ p \\ \frac{1}{2}p(u + a_{\frac{1}{2}}) \end{pmatrix}_L + \begin{pmatrix} 0 \\ \mathcal{P}^- p \\ \frac{1}{2}p(u - a_{\frac{1}{2}}) \end{pmatrix}_R \quad (12)$$

For $u > a$, $\mathbf{F}_{\frac{1}{2}} = \mathbf{F}_L$; For $u < -a$, $\mathbf{F}_{\frac{1}{2}} = \mathbf{F}_R$

Interface mass flux is introduced based on Wada-Liou AUSMD scheme:

$$(\rho u)_{\frac{1}{2}} = (\rho_L u_L^+ + \rho_R u_R^-) \quad (13)$$

$$u_L^+ = a_{\frac{1}{2}}\{\frac{M_L + |M_L|}{2} + \alpha_L[\frac{1}{4}(M_L + 1)^2 - \frac{M_L + |M_L|}{2}]\} \quad (14)$$

$$u_R^- = a_{\frac{1}{2}}\{\frac{M_R - |M_R|}{2} + \alpha_R[-\frac{1}{4}(M_R - 1)^2 - \frac{M_R - |M_R|}{2}]\} \quad (15)$$

The New E-CUSP Scheme, continued

Interface speed of sound

$$a_{\frac{1}{2}} = \frac{1}{2}(a_L + a_R) \tag{16}$$

$$M_L = \frac{u_L}{a_{\frac{1}{2}}}, \quad M_R = \frac{u_R}{a_{\frac{1}{2}}} \tag{17}$$

$$\alpha_L = \frac{2(p/\rho)_L}{(p/\rho)_L + (p/\rho)_R}, \quad \alpha_R = \frac{2(p/\rho)_R}{(p/\rho)_L + (p/\rho)_R} \tag{18}$$

Pressure splitting in momentum eq.

$$\mathcal{P}^{\pm} = \frac{1}{4}(M \pm 1)^2(2 \mp M) \pm \alpha M(M^2 - 1)^2, \quad \alpha = \frac{3}{16} \tag{19}$$

Numerical Dissipation

At stagnation $u = 0$, the dissipation of the new scheme:

$$\mathbf{D} = -\frac{a_{\frac{1}{2}}}{2} \begin{pmatrix} 0 \\ 0 \\ \delta p \end{pmatrix} \tag{20}$$

where

$$\delta p = p_R - p_L \tag{21}$$

The dissipation of the Roe scheme:

$$\mathbf{D}_{Roe} = -\frac{\tilde{a}_{\frac{1}{2}}}{2(\gamma - 1)} \begin{pmatrix} (\gamma - 1)/\tilde{a}^2_{\frac{1}{2}} \delta p \\ 0 \\ \delta p \end{pmatrix} \tag{22}$$

The dissipation of the new scheme is not greater than that of the Roe scheme.

The Sod Shock Tube Problem

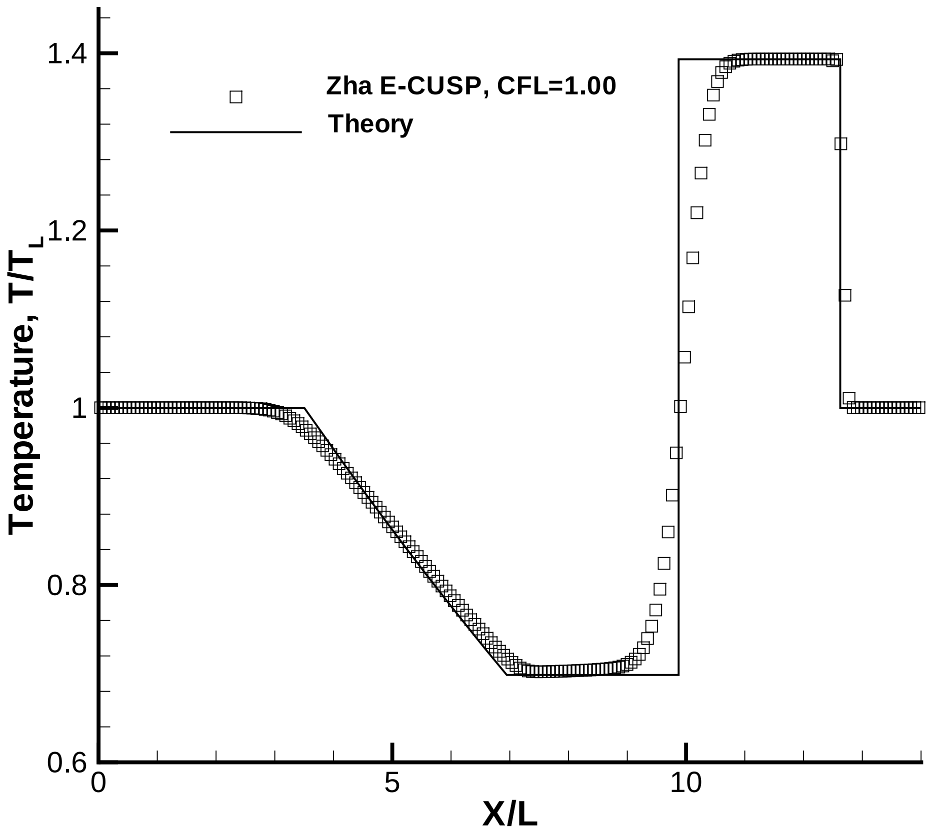


Figure 1: **Temperature, Zha E-CUSP scheme**

The Sod Shock Tube Problem

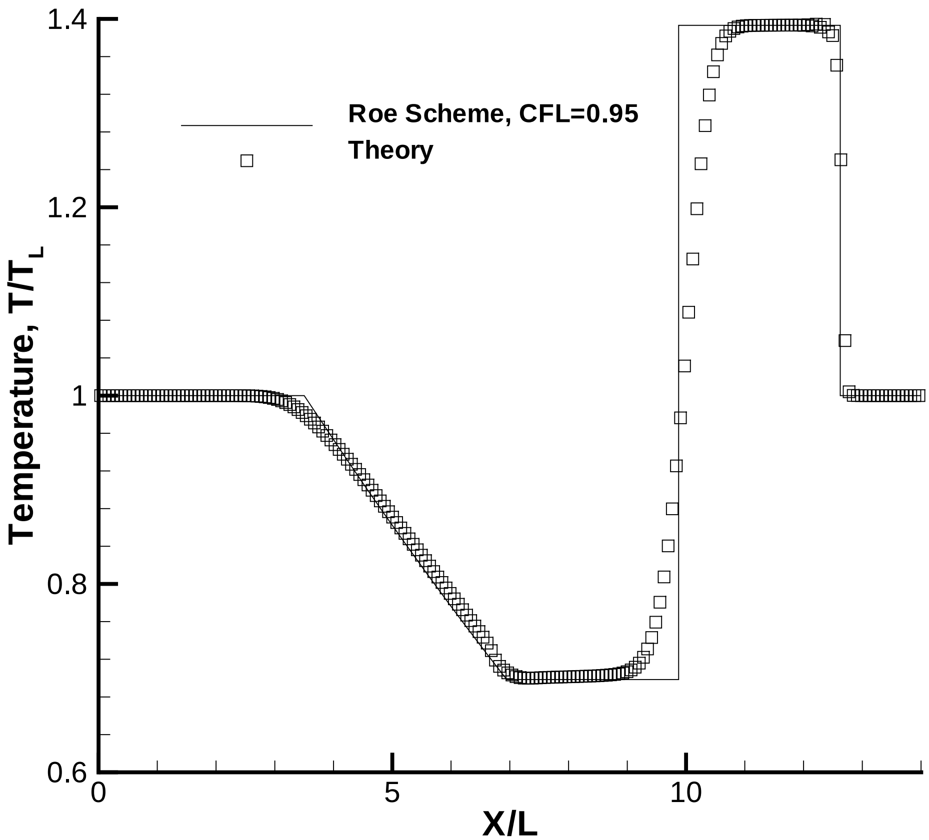


Figure 2: **Temperature, Roe scheme**

The Sod Shock Tube Problem

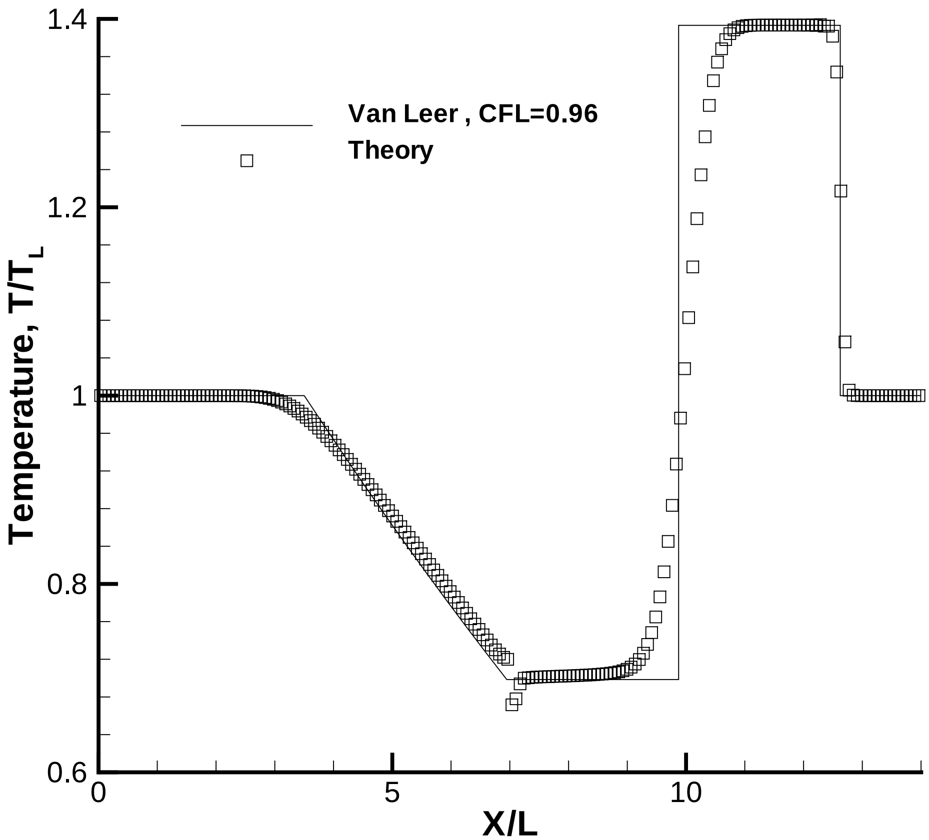


Figure 3: **Temperature, Van Leer scheme**

The Sod Shock Tube Problem

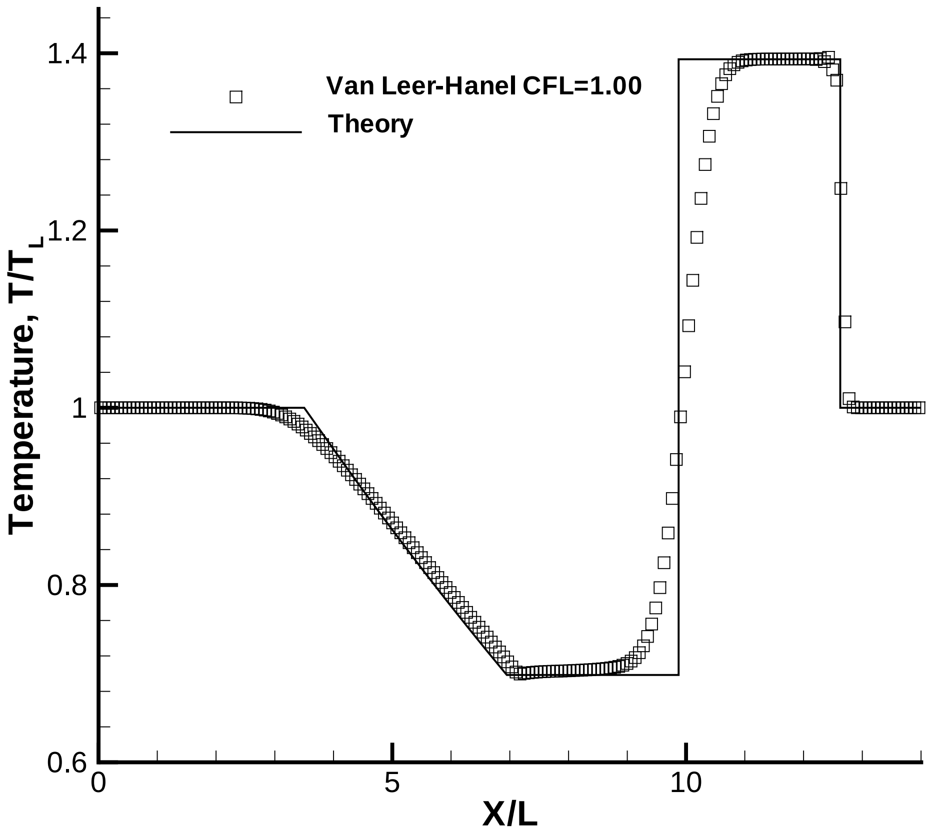


Figure 4: **Temperature, Van Leer-Hänel scheme**

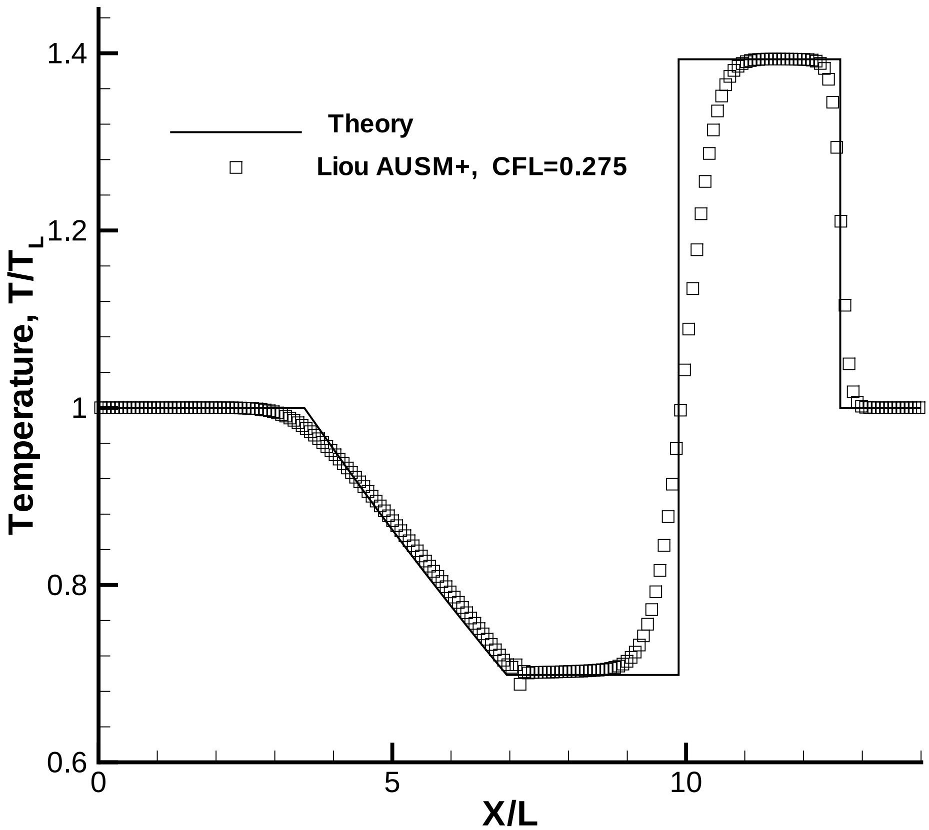


Figure 5: **Temperature, Liou AUSM$^+$ scheme**

The Sod Shock Tube Problem

Table 1: **Maximum CFL Numbers for Sod 1D Shock Tube**

Scheme	CFL Number
The new scheme (Zha CUSP)	1.00
Van Leer-Hänel	1.00
Van Leer	0.96
Roe	0.95
Liou $AUSM^+$	0.275

Slowly Moving Contact Surface

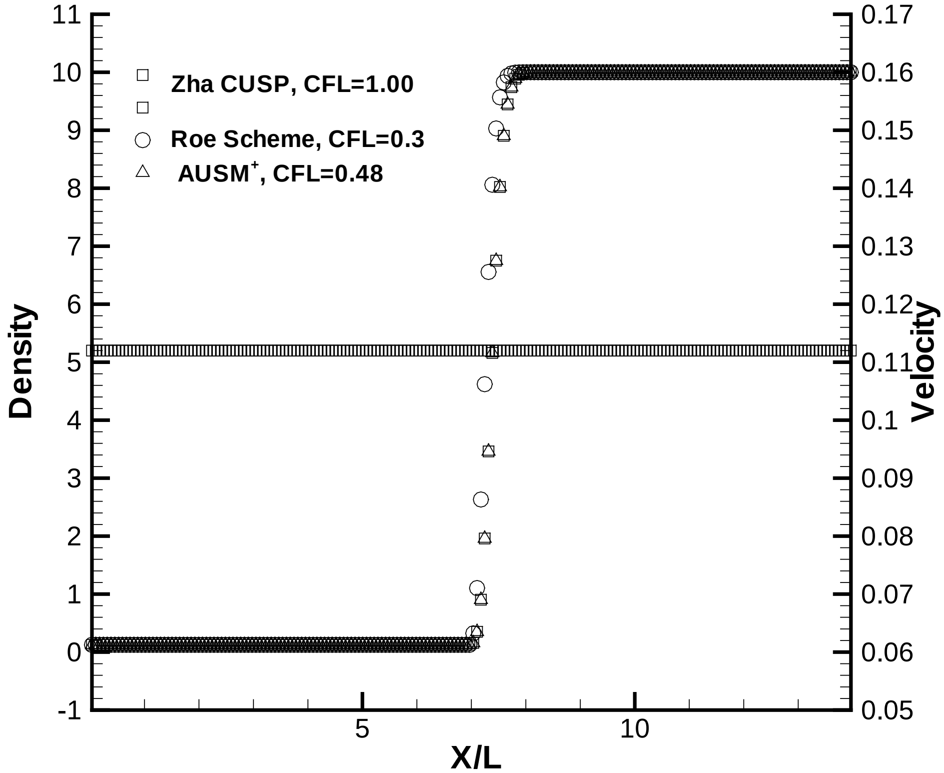


Figure 6: **Density and velocity, E-CUSP, Roe, AUSM$^+$ scheme**

Slowly Moving Contact Surface

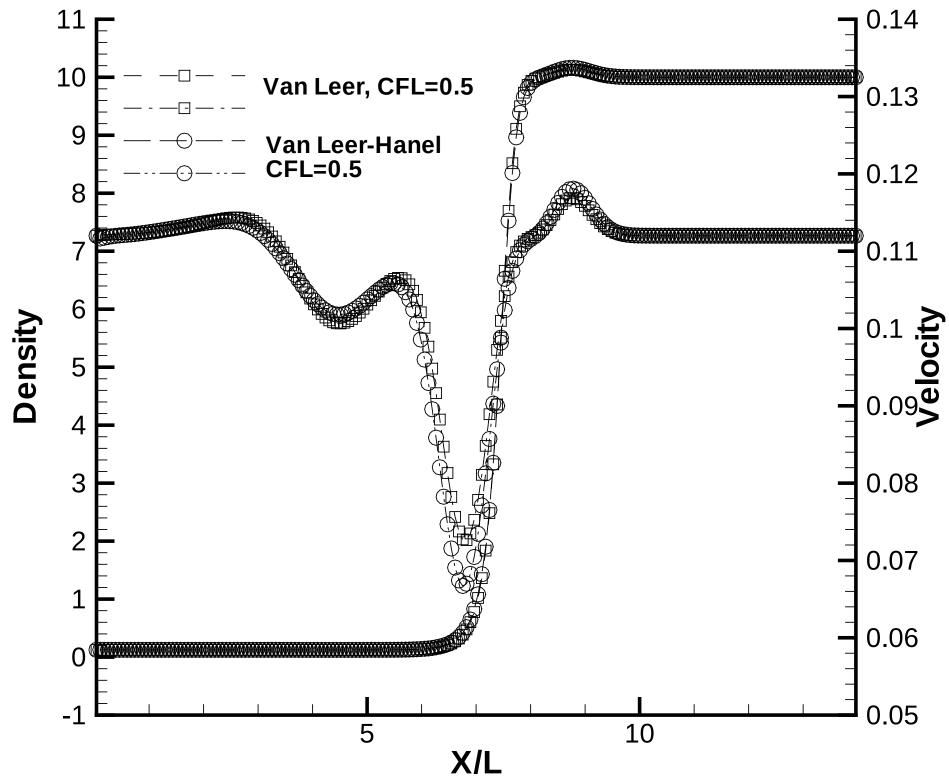
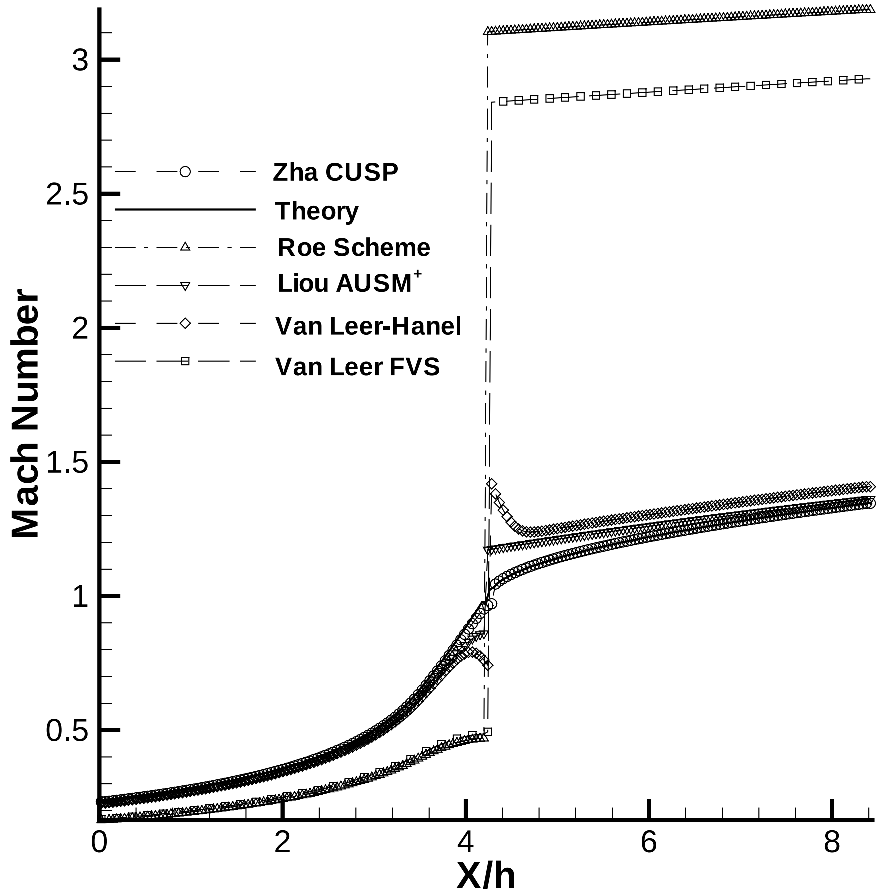
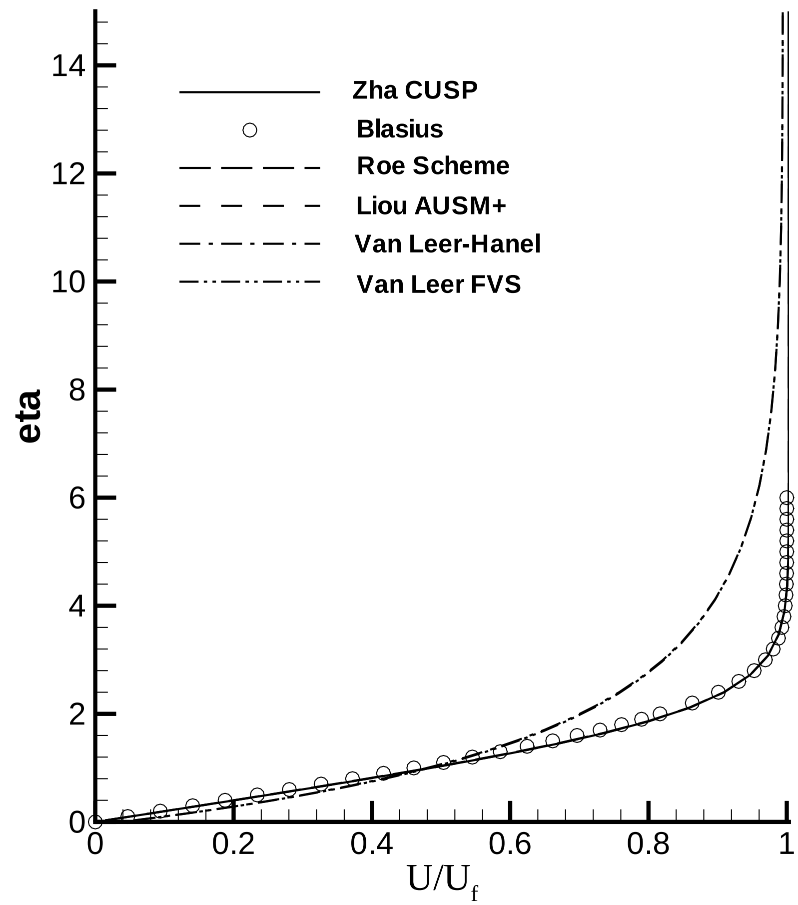


Figure 7: **Density and velocity, Van Leer, Van Leer-Hanel scheme**

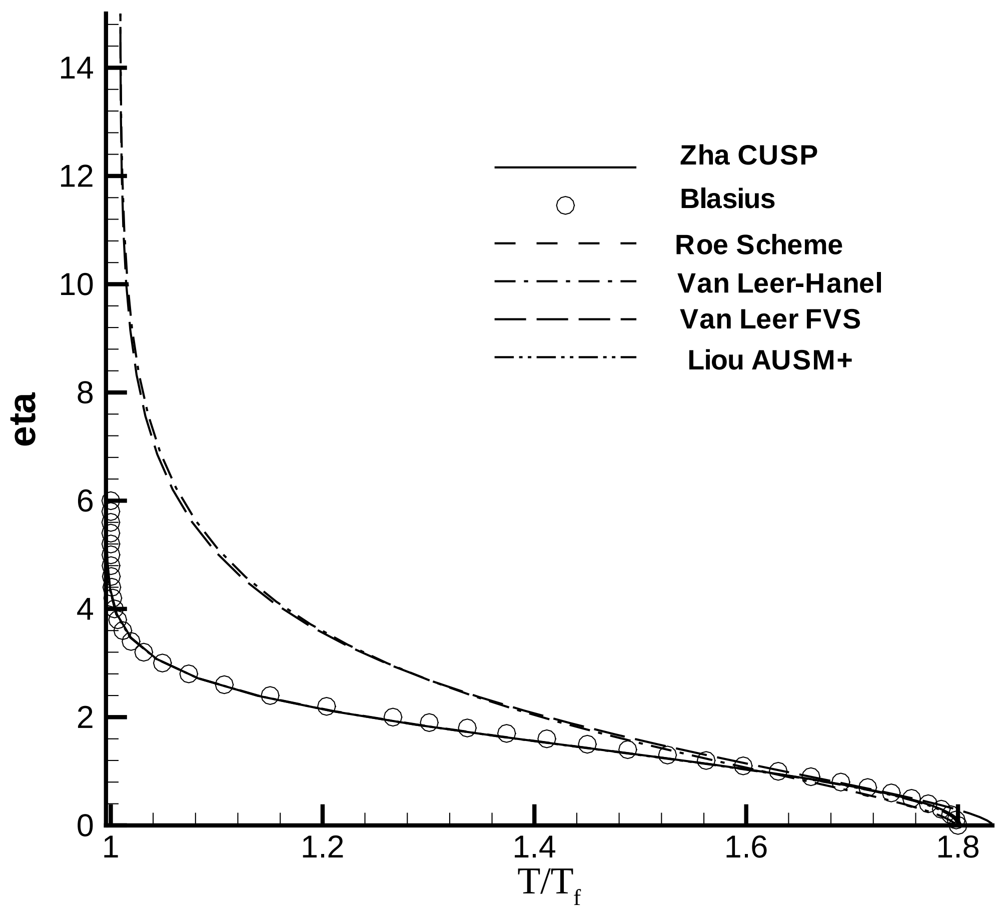
Quasi-1D Nozzle, Mach number



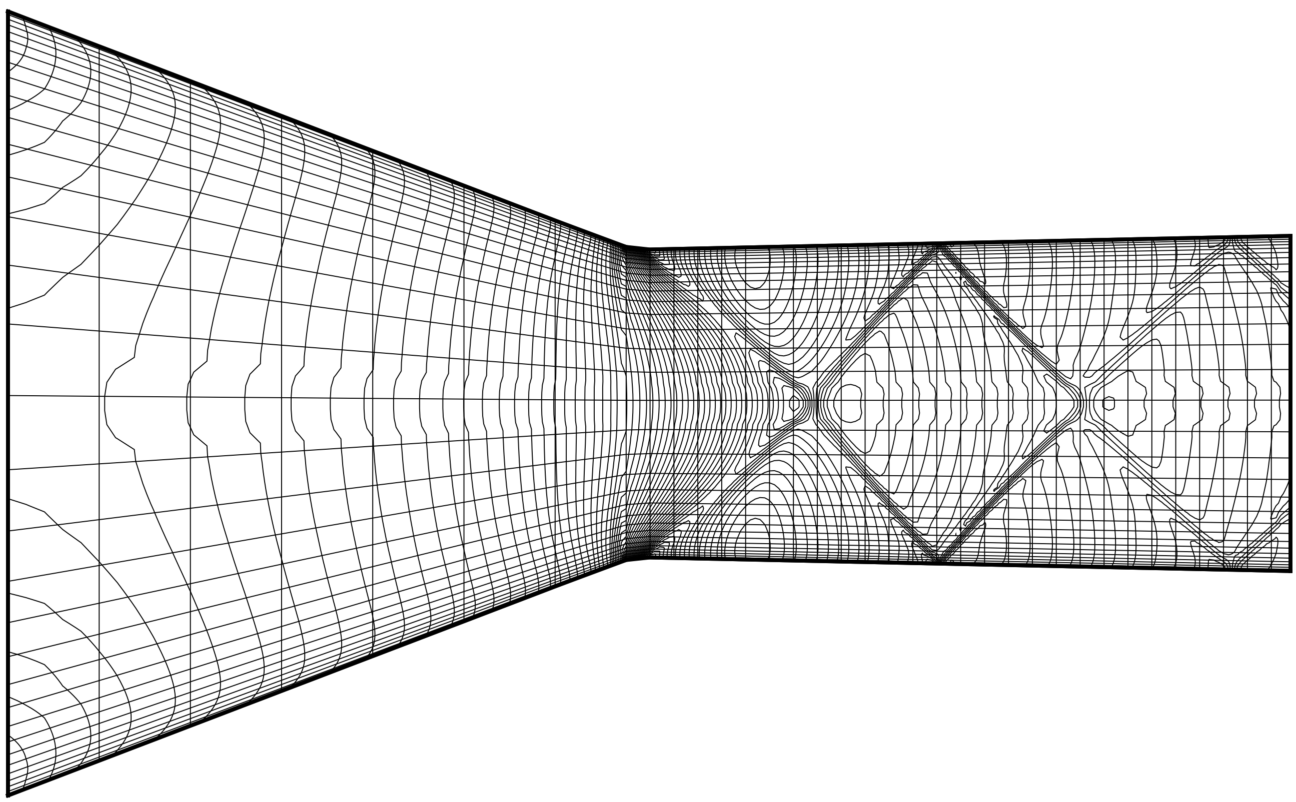
Laminar Flat Plate, M=2.0, Velocity Profile



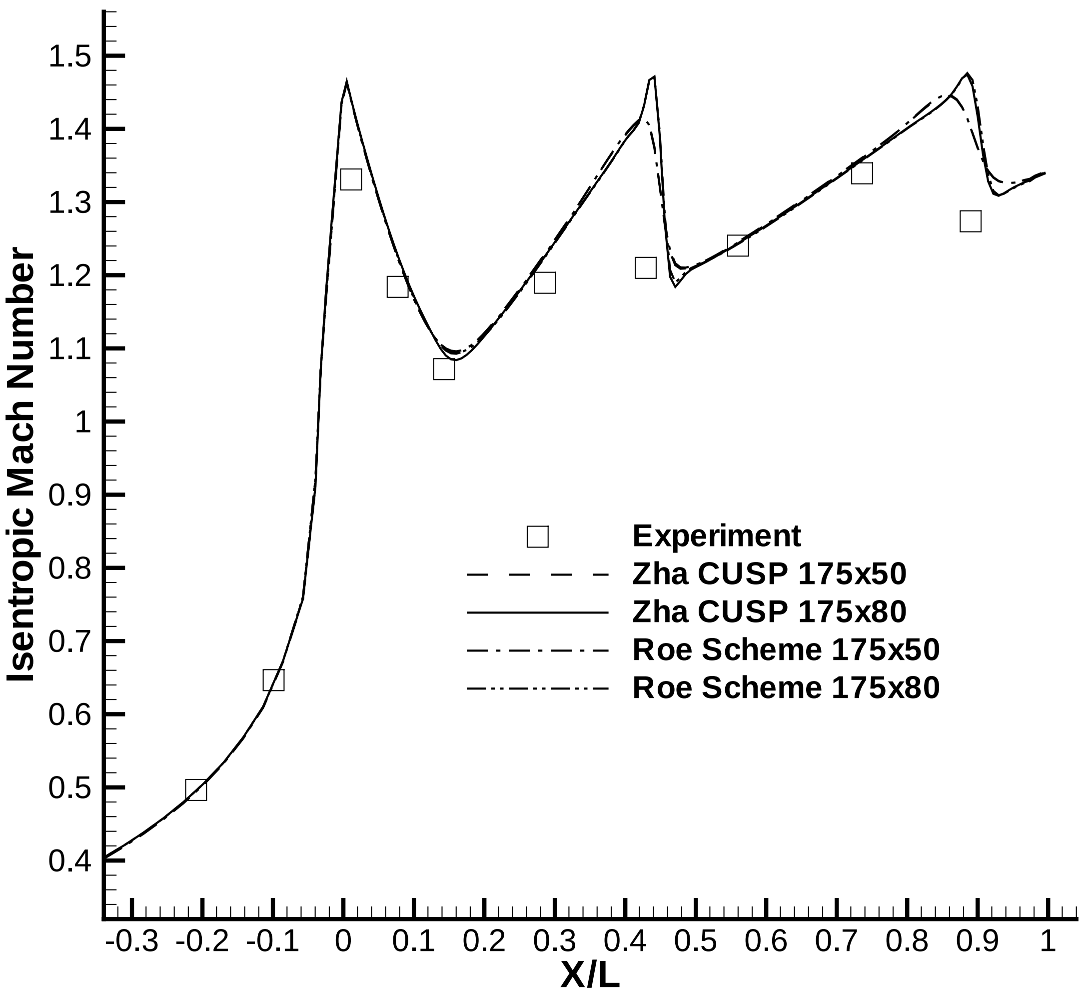
Laminar Flat Plate, M=2.0, Temperature Profile



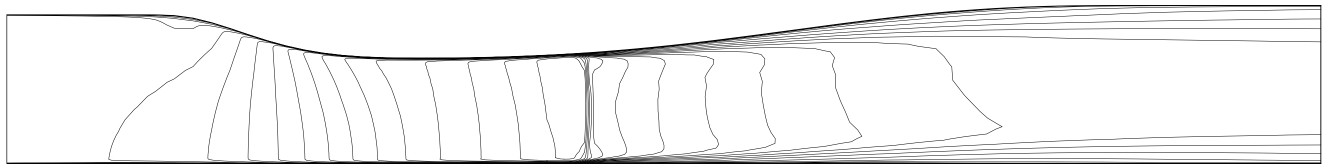
NASA Transonic Nozzle, Mach Number Contours, New E-CUSP Scheme

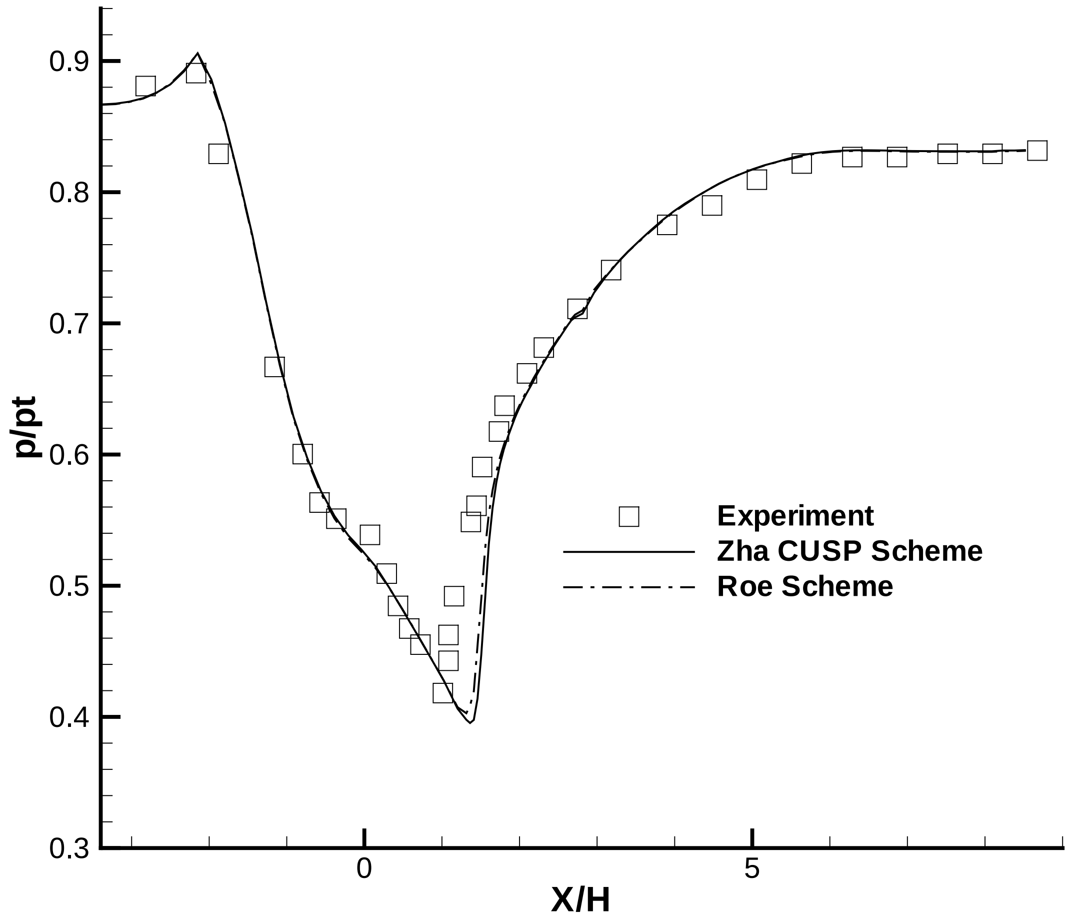


NASA Transonic Nozzle, Wall Mach Number Distribution

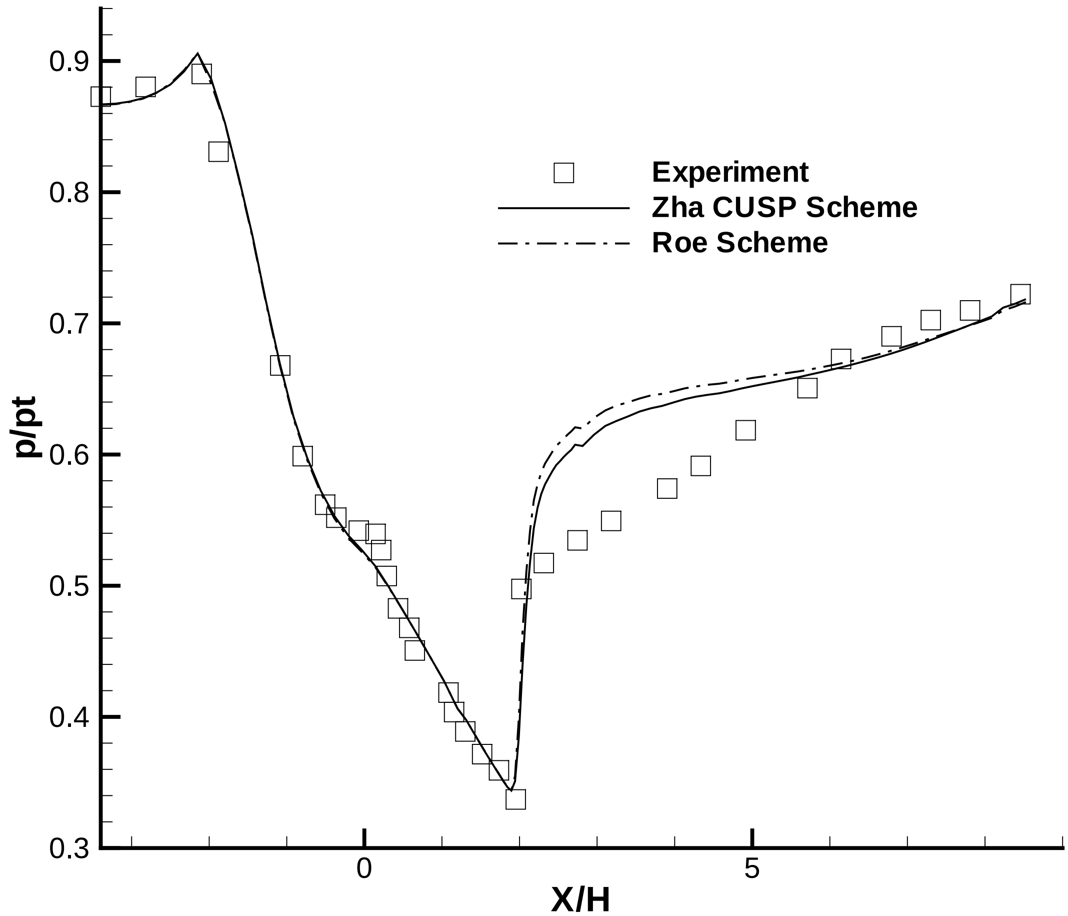


Inlet Diffuser, Mach Number Contours, $p_{out}/p_t = .83$





Inlet Diffuser, Surface Pressure Distribution, $p_{out}/p_t = .83$



Inlet Diffuser, Surface Pressure Distribution, $p_{out}/p_t = .72$

Inlet Diffuser, Mach Number Contours, $p_{out}/p_t = .72$

Zha CUSP Scheme

Roe Scheme

AUSM+ Scheme

Conclusions:

- The new E-CUSP scheme is efficient and has low numerical dissipation.

- Able to capture crisp shock profile and exact contact discontinuities

- For 1D Sod shock problem, $CFL_{max} = 1$, crispest shock profile

- For quasi-1D nozzle, no expansion shock generated at sonic point.

- For M=2 laminar flat plate, 1st order scheme obtains accurate velocity and temperature profiles

- For a transonic nozzle, oblique shock captured well

- For a transonic inlet-diffuser with shock wave/turbulent boundary layer interaction, the surface pressure agree well with experiment.